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W. P. Waddington
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# THE PRINCIPLES

# OF

# FLUXIONS:

## DESIGNED FOR THE USE OF STUDENTS IN THE UNIVERSITY.

BY WILLIAM DEALTRY, M.A.

PROFESSOR OF MATHEMATICS IN THE EAST-INDIA COLLEGE,
AND
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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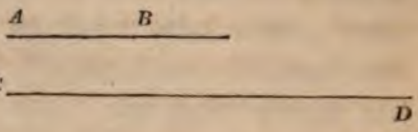
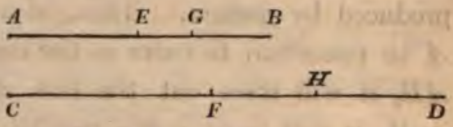


(4.) The velocities by which these quantities are produced may be either uniform or variable. If the velocities of two points $A$ and $C$ be uniform, and in the ratio of 1 : 2, the lines $AB$ and $CD$, described in the same time, will be in the same ratio of 1 : 2; and in this case the line $CD$ is said to *flow* with twice the velocity of $AB$; the increase of $CD$ in a given time is double of the increase of $AB$ in the same time, or the fluxion of $CD$ : the fluxion of $AB$ :: 2 : 1. And in general, if the uniform velocity of the point $A$ : the uniform velocity of the point $C$ :: $m : n$, the corresponding increments of $AB$ and $CD$ will be in the same ratio; that is, the fluxion of $AB$ : the fluxion of $CD$ :: $m : n$ or :: velocity of $A$ : velocity of $C$.



If the lines $AB$ and $CD$ are described with variable velocities, their corresponding increments are no longer proportional to the velocities, and therefore cannot represent the fluxions. The velocity of a body at any point of it's motion is represented by the space, which would be described in a given time with the velocity at that point continued uniform. Let $E$ and $F$ be two contemporaneous positions of $A$ and $C$; let $EG$ and $FH$ represent two spaces, which $A$ and $C$ would describe in the same time, if the velocities at $E$ and at $F$ were continued uniformly; and let $EB$ and $FD$ be the spaces which they actually describe by the variable velocities. The parts $GB$ and $HD$ are produced by accelerations which had no existence at $E$ and $F$, and are not described uniformly; whereas $EG$ and $FH$ are proportional to the velocities at $E$ and $F$, and are described in the same time. Hence the fluxion of $AE$ : fluxion of $CF$ :: $EG : FH$.

(5.) This reasoning may be illustrated by the doctrine of falling bodies. If a body descend by the force of gravity for 2″, it describes a space in the 2d second of 48 feet, nearly;

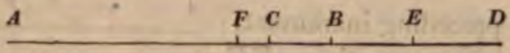
this may be divided into two parts; of which 32 feet are described by the velocity acquired at the end of the 1st second, and 16 by the acceleration of gravity. If then we should assume 48 as the measure of the velocity in the 2d second, the conclusion would be erroneous. Suppose a body to fall for 10″, the space described in the 11th second may be thus represented.

The velocity acquired in 10″, omitting fractions, = 320 feet; or the space described in the 11th second by
Feet.
that uniform velocity . . . . . . . . . . . . . = 320
The space in the 11th second, by gravity . . . = 16
∴ the whole space, from both causes . . . . . = 336

On this division of the spaces depends the whole method of fluxions.

(6.) When a quantity increases with a velocity which continually varies, the quantity, which measures the fluxion, is a limit between the preceding and succeeding increments, and is ultimately[*] equal to either of them.

1. Let the point *A* move on the straight line *AB* with a velocity perpetually increasing. Let *CB* represent the space described in a given time before *A* arrives at *B*; and let *BD* be the space described in the same time afterwards; both by the variable velocity. Suppose *BE* to be the space which



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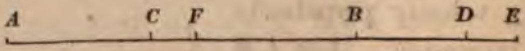
[*] The word *ultimately* is intended to denote that particular instant, when the time is diminished *sine limite.* Sir Isaac Newton thus describes ultimate velocity and ultimate ratios: "Per velocitatem ultimam intelligi eam, quâ corpus movetur, neque antequam attingit locum ultimum et motus cessat, neque postea, sed tunc cum attingit; id est, illam ipsam velocitatem quâcum corpus attingit locum ultimum, et quâcum motus cessat. Et similiter per ultimam rationem quantitatum evanescentium, intelligendam esse rationem quantitatum, non antequam evanescunt, non postea, sed quâcum evanescunt." Scholium Sect. Primæ.

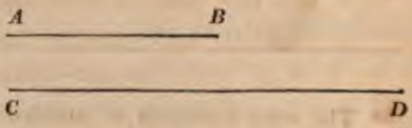
the point $A$ would describe in the same time, with the velocity at $B$ continued uniform; and in the line $BA$ take $BF = BE$. Then, since the velocity of $A$ is perpetually increasing, $BE$ is less than $BD$, and greater than $BC$; but $BE$ measures the fluxion; therefore the quantity which measures the fluxion is in this case greater than the preceding, and less than the succeeding increment.

Next, let the time of describing $CB$ or $BD$ be assumed extremely small; the difference between $BF$ and $BC$, or between $BE$ and $BD$, will ultimately be less than any assignable magnitude. For $ED$ is described by an acceleration, which was nothing at $B$; and $BE$ is described by an uniform finite velocity; therefore, by diminishing the time *sine limite*, $ED$ is indefinitely diminished with respect to $BE$, and $FC$ with respect to $BF$; or $BE$ is ultimately equal to $BD$, and $BC$ to $BF$; that is, the quantity which measures the fluxion is a limit between the increments, and ultimately equal either to the preceding or succeeding increment.

2. Let $AB$ be traced out by a velocity perpetually retarded. In this case, $BC$, the preceding increment, is greater than $BF$ or $BE$, and $BD$ is less; the reasoning and the conclusion are the same as before.

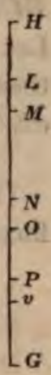


Cor. 1. If one quantity, as $AB$, increase uniformly, and another, as $CD$, increase with an accelerated or retarded velocity; the fluxion of $AB$ : fluxion of $CD$ :: increment of $AB$ in a given time : limit between the corresponding increments of $CD$.

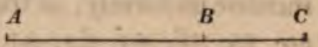


Cor. 2. Hence the fluxion of a quantity must have these two properties. 1. It must be a limit between the preceding and succeeding increments. 2. It must be proportional to the increment of a quantity, which flows uniformly.

(7.) In the preceding demonstration we have reasoned upon the hypothesis, that if the time be indefinitely small, $DE$ vanishes in respect of $BE$. This may be illustrated in the following manner. Suppose a projectile to be discharged in a direction $GH$ perpendicular to the horizon, and with a velocity of 1000 feet in a second, the force of gravity being considered as uniform, and the resistance of the air omitted. Take $GL=1000$ feet, $LM=16\frac{1}{12}$; then, at the end of $1''$, the body would be found at $M$, and $LM : LG :: 16 : 1000$ nearly $:: 2 : 125$. Next, take $GN=500$ feet, and $NO=4$ feet. Then, in half a second, while the body would uniformly describe $GN$, gravity would draw it through $NO$; therefore it will be found at $O$; and $NO : NG :: 4 : 500 :: 1 : 125$. Next, take $GP = 100$; then $Pv$, the space by gravity in $\frac{1}{10}''$, or whilst the body with an uniform motion would describe 100 feet, $= \frac{4}{25}$ of a foot; therefore in this case $Pv : PG :: \frac{4}{25} : 100 :: 4 : 2500 :: 1 : 625$. If we take the one thousandth part of a second, $Pv : PG ::$ $1 : 62{,}500$, nearly; so that in every case, as the time is diminished, the space through which gravity would draw the body bears a less ratio to the space described in that same time by the uniform finite velocity. And if the time be assumed indefinitely small, $Pv$ will ultimately become evanescent in respect of $PG$.

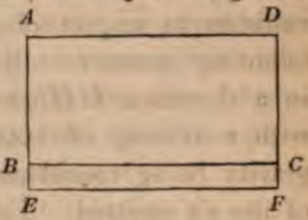


(8.) The first letters of the alphabet, $a$, $b$, $c$, $d$, &c. are generally assumed to represent invariable quantities; the last letters, as $w$, $x$, $y$, $z$, such as are variable. The fluxion of a simple quantity $x$ is denoted by a point over the letter, as $\dot{x}$. Thus, if $AB$ is equal to $x$, $BC$, the fluxion of $AB$, $=\dot{x}$.



(9.) If the fluxion of $x$ be expressed by $\dot{x}$, the fluxion of $ax$ will be $a\dot{x}$.

For if $x$ increase uniformly, $ax$ will also increase uniformly, and its velocity of increase will be $a$ times greater than that of $x$, that is, it will $=a\dot{x}$. Let $ABCD$ be a parallelogram, of which the side $AD$ remains constant, while $AB$ flows uniformly. $BCFE$ will represent its fluxion. Let $AD=a$, $AB=x$, $BE=\dot{x}$; then $ABCD=ax$, and $BCFE=a\dot{x}$.



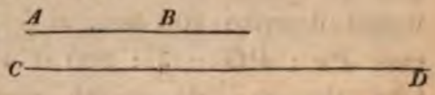
(10.) If two quantities have to each other a given ratio, their fluxions are in the same given ratio.

$$\text{Let } x : y :: a : b,$$
$$\text{then } bx=ay;\quad \therefore\ b\dot{x}=a\dot{y};$$
$$\therefore\ \dot{x} : \dot{y} :: a : b.$$

(11.) The fluxion of $a \pm x$ is $\pm\dot{x}$.

Let $AB$, a constant line, $=a$,

$CD$, a variable line, $=x$.



The line $AB$ does not affect the increase or decrease of $x$; so that the variation of $AB+CD$ is the same as the variation of $CD$ alone; that is, the fluxion of $a \pm x = \pm\dot{x}$.

Cor. Constant quantities, connected with variable ones by the sign + or −, disappear when the fluxion is taken.

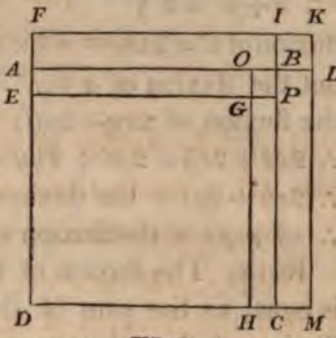
(12.) If any numerical or algebraical quantity $x$ be supposed to increase uniformly, the squares of the succeeding quantities will increase with velocities continually accelerated.

Let the numbers 5, 6, 7, 8, 9, 10, be assumed, which increase uniformly, as the several differences $=1$. Their squares are 25, 36, 49, 64, 81, 100, of which the several differences are 11, 13, 15, 17, 19; therefore the numbers do not increase uniformly.

In general, let $x$ represent any algebraical quantity, and $x'$ it's increment; then the present and succeeding values are $x$, $x+x'$, $x+2x'$, $x+3x'$, &c.; and the present and succeeding values of the squares are $x^2$, $\overline{x+x'}|^2$, $\overline{x+2x'}|^2$, $\overline{x+3x'}|^2$, &c.

The increments are obtained by subtracting in this case each preceding value from that which follows it; therefore the increments are,$\overset{2xx'+x'^2}{\wedge}$ $2xx'+3x'^2$, $2xx'+5x'^2$ &c.; that is, if the quantities themselves increase with an uniform velocity, their squares increase with a velocity perpetually accelerated. The same reasoning is true of the higher powers.

Cor. It is manifest, that if $2xx'$ were equal to the difference between these several squares, their velocity of increase would be uniform; so that the parts $x'^2$, $3x'^2$, $5x'^2$, &c. are the effects of acceleration; whence (by Art. 3. and 4.) these quantities, which involve the powers of $x'$, are to be omitted in taking the fluxions.



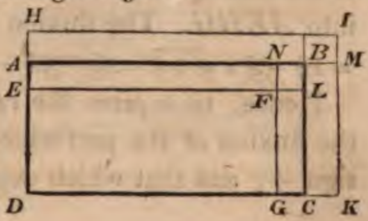
(13.) To find the fluxion of $x^2$. Let $AB=x$; $ABCD=x^2$; let $AE$ and $AF$ each = the increment of $x=x'$; and let $EGHD$, $FKMD$ represent the preceding and succeeding values of $ABCD$. Then the gnomon $APH$, or the preceding increment, $=AP+BH-BG$ $=2xx'-x'^2$; and the succeeding increment, $FLC=2xx'+x'^2$. Now the limit between these is $2xx'$, or ultimately $2x\dot{x}$; $\therefore$ the fluxion of $x^2=2x\dot{x}$.

The same result is obtained by the following process. Let $x-x'$, $x$, and $x+x'$, represent the preceding, present, and succeeding values of $x$. Their squares are

$$x^2-2xx'+x'^2,\ x^2,\ \text{and}\ x^2+2xx'+x'^2;$$

$\therefore$ the preceding increment $=2xx'-x'^2$, and the succeeding . . . . . . $=2xx'+x'^2$, of which the limit is $2xx'$, or the fluxion $=2x\dot{x}$.

(14.) To find the fluxion of a rectangle $xy$.



Let $ABCD$ be the rectangle; $AD=x$, $AB=y$. Take $AE$, $AH$, each $=x'$, and $BN$, $BM$, each $=y'$, and complete the rectangles $EFGD$, $HIKD$. These two rectangles will represent the preceding and succeeding values of $ABCD$.

The preceding increment = the gnomon $ALG = AL + BG - BF$,

or . . . = . . . $yx' + xy' - y'x'$;

and the succeeding increment $= HB + BK + IB$,

or . . . = . . . $yx' + xy' + y'x'$;

and the limit between them is $yx' + xy'$, or the fluxion is $y\dot{x} + x\dot{y}$.

The same result may be obtained in the following manner.

$$\overline{x+y}|^2 = x^2 + 2xy + y^2.$$

$\left.\begin{array}{l} \text{Let } x+y = z \\ \therefore \dot{x}+\dot{y} = \dot{z} \end{array}\right\}$ then $\overline{x+y}|^2 = z^2$; $\therefore$ the fluxion of $\overline{x+y}|^2$ = the fluxion of $z^2 = 2z\dot{z}$, or $= 2 \times \overline{x+y} \times \overline{\dot{x}+\dot{y}} = 2x\dot{x} + 2x\dot{y} + 2y\dot{x} + 2y\dot{y}$; but the fluxion of $\overline{x+y}|^2 =$ the fluxion of $x^2 + 2xy + y^2 = 2x\dot{x} +$ the fluxion of $2xy + 2y\dot{y}$;

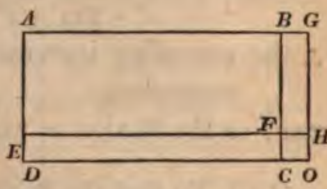
$\therefore 2x\dot{x} + 2x\dot{y} + 2y\dot{x} + 2y\dot{y} = 2x\dot{x} +$ the fluxion of $2xy + 2y\dot{y}$;

$\therefore 2x\dot{y} + 2y\dot{x} =$ the fluxion of $2xy$, or $= 2 \times$ fluxion of $xy$;

$\therefore x\dot{y} + y\dot{x} =$ the fluxion of $xy$.

RULE. The fluxion of the product of two flowing quantities is equal to the sum of the product of each quantity and the fluxion of the other.

(15.) In this Article, $x$ and $y$ are both supposed to increase; for $\dot{x}$ and $\dot{y}$ are both assumed positive. But if one of them, as $x$, decrease, whilst the other continues to increase, the fluxion is $x\dot{y} - y\dot{x}$.



Let $ABCD$ be a parallelogram, of which the side $AB$ increases, and $AD$ decreases, with an uniform velocity; take $AB = y$, $AD = x$; $BG = \dot{y}$, $DE = \dot{x}$; and by this variation of the sides, let $ABCD$ be changed into $AEHG$. The fluxion of $ABCD = BH - EC = BO - EO = x\dot{y} - \dot{x} \times \overline{y+\dot{y}} = x\dot{y} - y\dot{x} - \dot{x}\dot{y}$ = ultimately, $x\dot{y} - y\dot{x}$.

Hence, to express the rate at which any quantity increases, the fluxion of the part which increases must be written with the sign +, and that which decreases with the sign −.

If a negative quantity increase, it must be considered as a decreasing positive quantity, and its fluxion is $-$.

(16.) To find the fluxion of $xyz$.
Let $xy=v$; then $xyz=vz$; $\therefore$ the fluxion of $xyz=$ fluxion of $vz$ $=v\dot{z}+z\dot{v}$; but $\dot{v}=x\dot{y}+y\dot{x}$, $\therefore$ the fluxion of $xyz=xy\dot{z}+zx\dot{y}$ $+zy\dot{x}$.

Rule. Hence the fluxion of the product of any number of flowing quantities is obtained by taking the sum of the products, which arise from multiplying together the fluxion of each quantity, and the product of all the others.

(17.) To find the fluxion of any power of a simple quantity $x^n$.
Let $x-x'$, $x$, and $x+x'$, be the preceding, present, and succeeding values of $x$. Then the corresponding values of $x^n$ are,

$$\overline{x-x'}|^{n}, \text{ or } x^n-nx^{n-1}x'+n\,.\frac{n-1}{2}\;x^{n-2}x'^2-\text{ \&c.}$$

$$x^n, \text{ or } x^n,$$

$$\text{and } \overline{x+x'}|^{n}, \text{ or } x^n+nx^{n-1}x'+n\,.\frac{n-1}{2}\;x^{n-2}x'^2+\text{ \&c.}$$

$$\therefore \text{ the preceding increment } =nx^{n-1}x'-n\,.\frac{n-1}{2}\;x^{n-2}x'^2+\text{ \&c.}$$

$$\text{the succeeding } \ldots\ldots =nx^{n-1}x'+n\,.\frac{n-1}{2}\;x^{n-2}x'^2-\text{ \&c.}$$

and the limit $=nx^{n-1}x'$, or the fluxion $=nx^{n-1}\dot{x}$.

In this case, $n$ may be either positive or negative, a whole number or a fraction.

(18.) To find the fluxion of any power of a compound quantity, as $\overline{a^m+x^m}|^{\frac{p}{q}}$.

$$\text{Let } \overline{a^m+x^m}|^{\frac{p}{q}}=y, \;\therefore\; a^m+x^m=y^{\frac{q}{p}},$$

$$\therefore\; mx^{m-1}\dot{x}=\frac{q}{p}\times y^{\frac{q}{p}-1}\,\dot{y},$$

$$\therefore\; \dot{y}=\frac{p}{q}\times\frac{mx^{m-1}\dot{x}}{y^{\frac{q}{p}-1}}=\frac{p}{q}\times\frac{mx^{m-1}\dot{x}}{\overline{a^m+x^m}|^{\frac{p}{q}\times\overline{\frac{q}{p}-1}}}$$

$$=\frac{p}{q}\times\frac{mx^{m-1}\dot{x}}{\overline{a^m+x^m}|^{1-\frac{p}{q}}}=\frac{p}{q}\times mx^{m-1}\dot{x}\times\overline{a^m+x^m}|^{\frac{p}{q}-1}.$$

that is, the fluxion of $\overline{a^m+x^m}|^{\frac{p}{q}} = \frac{p}{q} \times mx^{m-1}\dot{x} \times \overline{a^m+x^m}|^{\frac{p}{q}-1}$.

In cases of this kind $\overline{a^m+x^m}$ is called the root, and the fluxion of this root is $mx^{m-1}\dot{x}$.

Rule. Hence the fluxion of any power of a flowing quantity, whether simple or compound, is found by multiplying together the index, the next inferior power, and the fluxion of the root.

(19.) To find the fluxion of a fraction, as $\frac{x}{y}$.

Let $\frac{x}{y} = z$, $\therefore x = yz$, $\therefore \dot{x} = y\dot{z} + z\dot{y}$, $\therefore y\dot{z} = \dot{x} - z\dot{y} = \dot{x} - \frac{x\dot{y}}{y} = \frac{y\dot{x} - x\dot{y}}{y}$, $\therefore \dot{z} = \frac{y\dot{x} - x\dot{y}}{y^2}$. But $\dot{z}$ = the fluxion of $\frac{x}{y}$, $\therefore$ the fluxion of $\frac{x}{y} = \frac{y\dot{x} - x\dot{y}}{y^2}$.

Hence the Rule. From the fluxion of the numerator, multiplied into the denominator, subtract the fluxion of the denominator multiplied into the numerator, and divide by the square of the denominator.

(20.) Examples of the different Rules.

Ex. 1. The fluxion of $x^8 = 8x^7\dot{x}$.

Ex. 2. The fluxion of $x^9 = 9x^8\dot{x}$.

Ex. 3. The fluxion of $\overline{a+x}|^6 = 6\dot{x} \times \overline{a+x}|^5$.

Ex. 4. The fluxion of $\overline{2ax-x^2}|^2 = \overline{4a\dot{x}-4x\dot{x}} \times \overline{2ax-x^2}$.

Ex. 5. The fluxion of $\overline{a^2-x^2}|^{\frac{4}{5}} = \frac{4}{5} \times -2x\dot{x} \times \overline{a^2-x^2}|^{-\frac{1}{5}} = \frac{-8x\dot{x}}{5 \times \overline{a^2-x^2}|^{\frac{1}{5}}}$.

Ex. 6. The fluxion of $\overline{ax^2+bx^3-cx^4}|^{\frac{3}{2}} = \overline{3ax\dot{x}+\frac{9bx^2\dot{x}}{2} - 6cx^3\dot{x}} \times \overline{ax^2+bx^3-cx^4}|^{\frac{1}{2}}$.

Ex. 7. The fluxion of $x^3y^4 = 3x^2\dot{x} \times y^4 + 4y^3\dot{y} \times x^3$
$= 3y^4x^2\dot{x} + 4x^3 \times y^3\dot{y}$.

Ex. 8. The fluxion of $x^2y^3z^{\frac{5}{3}} = 2y^3z^{\frac{5}{3}}x\dot{x} + 3x^2z^{\frac{5}{3}}y^2\dot{y} + \frac{5}{3} \times x^2y^3z^{\frac{2}{3}}\dot{z}$.

Ex. 9. The fluxion of $x \times \overline{a^2+x^2}|^{\frac{3}{2}} = \dot{x} \times \overline{a^2+x^2}|^{\frac{3}{2}} + 3x\dot{x} \times \overline{a^2+x^2}|^{\frac{1}{2}} \times x = \overline{a^2\dot{x} + x^2\dot{x} + 3x^2\dot{x}} \times \overline{a^2+x^2}|^{\frac{1}{2}} = \overline{a^2\dot{x} + 4x^2\dot{x}} \times \overline{a^2+x^2}|^{\frac{1}{2}}$.

Ex. 10. The fluxion of $\overline{x^2+y^2}|^{\frac{3}{2}} \times \overline{x^2-y^2}|^{\frac{4}{3}} = \overline{3x\dot{x}+3y\dot{y}} \times \overline{x^2+y^2}|^{\frac{1}{2}} \times \overline{x^2-y^2}|^{\frac{4}{3}} + \frac{8x\dot{x} - 8y\dot{y}}{3} \times \overline{x^2-y^2}|^{\frac{1}{3}} \times \overline{x^2+y^2}|^{\frac{3}{2}}$.

Ex. 11. The fluxion of $\overline{a^2+x^2}|^{\frac{1}{2}} \times \overline{b^2-y^2}|^{\frac{3}{2}} = x\dot{x} \times \overline{a^2+x^2}|^{-\frac{1}{2}} \times \overline{b^2-y^2}|^{\frac{3}{2}} - 3y\dot{y} \times \overline{a^2+x^2}|^{\frac{1}{2}} \times \overline{b^2-y^2}|^{\frac{1}{2}} = \frac{x\dot{x} \times \overline{b^2-y^2}|^{\frac{3}{2}}}{\overline{a^2+x^2}|^{\frac{1}{2}}} - 3y\dot{y} \times \overline{a^2+x^2}|^{\frac{1}{2}} \times \overline{b^2-y^2}|^{\frac{1}{2}}$.

Ex. 12. The fluxion of $\frac{x^2}{y^2} = \frac{2x\dot{x} \times y^2 - 2y\dot{y} \times x^2}{y^4} = \frac{2y^2x\dot{x} - 2x^2y\dot{y}}{y^4}$.

Ex. 13. The fluxion of $\frac{1}{x^6} = \frac{0 - 6x^5\dot{x}}{x^{12}} = \frac{-6x^5\dot{x}}{x^{12}} = \frac{-6\dot{x}}{x^7}$.

Ex. 14. The fluxion of $\frac{a}{x^n} = \frac{0 - nx^{n-1}\dot{x} \times a}{x^{2n}} = \frac{-na\dot{x}}{x^{n+1}}$.

Ex. 15. The fluxion of $\frac{x-y}{x^3} = \frac{\overline{\dot{x}-\dot{y}} \times x^3 - 3x^2\dot{x} \times \overline{x-y}}{x^6} = \frac{x^3\dot{x} - x^3\dot{y} - 3x^3\dot{x} + 3yx^2\dot{x}}{x^6} = \frac{3yx^2\dot{x} - 2x^3\dot{x} - x^3\dot{y}}{x^6}$.

Ex. 16. The fluxion of $\frac{x}{\overline{a+x}|^5}$ = fluxion of $x \times \overline{a+x}|^{-5} = \dot{x} \times \overline{a+x}|^{-5} - 5\dot{x} \times \overline{a+x}|^{-6} \times x = \overline{a\dot{x} + x\dot{x} - 5x\dot{x}} \times \overline{a+x}|^{-6} = \frac{a\dot{x} - 4x\dot{x}}{\overline{a+x}|^6}$.

the flux of

Ex. 17. The fluxion of $\sqrt{\frac{a+x}{b+y}} = \overline{a+x}|^{\frac{1}{2}} \times \overline{b+y}|^{-\frac{1}{2}} =$

$\frac{1}{2}\dot{x} \times \overline{a+x}|^{-\frac{1}{2}} \times \overline{b+y}|^{-\frac{1}{2}} - \frac{1}{2}\dot{y} \times \overline{a+x}|^{\frac{1}{2}} \times \overline{b+y}|^{-\frac{3}{2}} =$

$$\frac{\dot{x}}{2 \times \overline{a+x}|^{\frac{1}{2}} \times \overline{b+y}|^{\frac{1}{2}}} - \frac{\dot{y} \times \overline{a+x}|^{\frac{1}{2}}}{2 \times \overline{b+y}|^{\frac{3}{2}}}.$$

Ex. 18. The fluxion of $\frac{\sqrt{a^3+x^3}}{\overline{a^2+y^2}|^{\frac{1}{3}}}$ = fluxion of $\overline{a^3+x^3}|^{\frac{1}{2}}$

$\times \overline{a^2+y^2}|^{-\frac{1}{3}} = \frac{3x^2\dot{x}}{2} \times \overline{a^3+x^3}|^{-\frac{1}{2}} \times \overline{a^2+y^2}|^{-\frac{1}{3}} - \frac{2y\dot{y}}{3} \times \overline{a^2+y^2}|^{-\frac{4}{3}}$

$$\times \overline{a^3+x^3}|^{\frac{1}{2}} = \frac{3x^2\dot{x}}{2 \times \overline{a^3+x^3}|^{\frac{1}{2}} \times \overline{a^2+y^2}|^{\frac{1}{3}}} - \frac{2y\dot{y} \times \overline{a^3+x^3}|^{\frac{1}{2}}}{3 \times \overline{a^2+y^2}|^{\frac{4}{3}}}.$$

Ex. 19. The fluxion of $\frac{x-1}{\sqrt{2x-x^2}}$, or of $\overline{x-1} \times \overline{2x-x^2}|^{-\frac{1}{2}} =$

$$\frac{\dot{x} \times \overline{2x-x^2}}{\overline{2x-x^2}|^{\frac{3}{2}}} + \frac{\overline{x\dot{x}-\dot{x}} \times \overline{x-1}}{\overline{2x-x^2}|^{\frac{3}{2}}} = \frac{\dot{x}}{\overline{2x-x^2}|^{\frac{3}{2}}}.$$

Ex. 20. The fluxion of $-\frac{x+1}{\sqrt{2x+x^2}} = \frac{\dot{x}}{\overline{2x+x^2}|^{\frac{3}{2}}}.$

Ex. 21. The fluxion of $\frac{-\sqrt{a^2-x^2}}{a^2x} = \frac{\dot{x}}{x^2 \times \sqrt{a^2-x^2}}.$

Ex. 22. The fluxion of $\frac{-\overline{a^2+x^2}|^{\frac{3}{2}}}{3a^2x^3} = \frac{\dot{x} \times \sqrt{a^2+x^2}}{x^4}.$

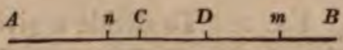
Ex. 23. The fluxion of $\frac{y^2}{\sqrt{y^2+b^2}}$ = fluxion of $y^2 \times \overline{y^2+b^2}|^{-\frac{1}{2}}$

$$= 2y\dot{y} \times \overline{y^2+b^2}|^{-\frac{1}{2}} - \frac{y\dot{y} \times y^2}{\overline{y^2+b^2}|^{\frac{3}{2}}} = \frac{2y^3\dot{y} + 2b^2y\dot{y} - y^3\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}} = \frac{y^3\dot{y} + 2b^2y\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}}.$$

## CHAP. II.

### ON THE MAXIMA AND MINIMA OF QUANTITIES.

(21.) THE fluxion of a quantity, when it is a maximum or minimum, $=0$.



Let two points, $m$ and $n$, begin to move on the straight line $AB$ at the same time; $m$ from the point $C$ with a given uniform velocity, and $n$ from $A$ with a velocity less than that of $m$ at the commencement of the motion, but which is continually increasing. Let $D$ be the point, in which the velocity of $n$ equals that of $m$; it is evident that the distance $nm$ is perpetually increasing till $n$ arrives at $D$, and then $n$ begins to overtake $m$, or $nm$ decreases. At $D$ therefore the distance between $n$ and $m$ neither increases nor decreases; and consequently its fluxion $= 0$. But in this case $nm$ is a maximum; hence the fluxion of a quantity, when it is a maximum, $= 0$.

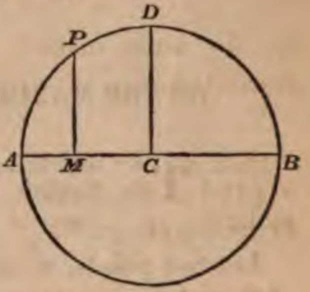
Next, let the velocity of $n$ at first, be greater than that of $m$ at first, but perpetually decrease; $nm$ continually decreases, till the velocity of $n$ is equal to that of $m$, and afterwards it increases. Let $D$ be the point, in which the velocity of $n$ is equal to that of $m$; here then $nm$ is neither increasing nor decreasing; therefore its fluxion is nothing; but here $nm$ is a minimum; consequently the fluxion of a quantity, which is a minimum, $=0$.

(22.) If a quantity be a maximum or a minimum, any power or root of that quantity is a maximum or a minimum; for the increase or decrease of the power or root will depend upon the increase or decrease of the original quantity.

In the same manner any constant multiple, or part of the original quantity, is at the same time a maximum or minimum.

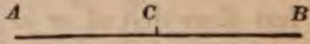
## (23.) Examples.

Ex. 1. To find the greatest ordinate in a given circle.



Let $AM = x$; $AB = 2a$; $BM = 2a - x$; then $MP^2 = 2ax - x^2$ a maximum; $\therefore\ 2a\dot{x} - 2x\dot{x} = 0$, and $x = a$; that is, if $C$ be the centre, and $CD$ be drawn perpendicular to $AB$, it is the ordinate required.

Ex. 2. To divide a given line $AB$ into two parts $x$ and $y$, so that the rectangle $xy$ may be a maximum.



Let $AC = x$; $CB = y$; $AB = a$; then $x + y = a$, and $\dot{x} + \dot{y} = 0$; $\therefore\ \dot{x} = -\dot{y}$; also $xy$ is a maximum; $\therefore\ x\dot{y} + y\dot{x} = 0$, or substituting $-\dot{y}$ for $\dot{x}$, $-x\dot{x} + y\dot{x} = 0$; hence $x = y$, or $AB$ must be bisected in $C$.

Cor. Hence to divide a quantity $a$ into three parts $x$, $y$, and $z$, so that their continued product may be a maximum, the parts must be equal. For if $x$ be assumed constant, the product $yz$, and therefore $xyz$ is a maximum, when $y = z$; if $y$ be assumed constant, the product is a maximum, when $x = z$; if $z$ be constant, $x$ must equal $y$. And in the same manner, into whatever number of parts a given line is divided, they are all equal, when their continued product is a maximum.

Ex. 3. To divide a given line $AB$ into two parts $x$ and $y$, so that $x^m \times y^n$ may be a maximum.

Let $AB = a$; $AC = x$; $CB = y$. Then $x + y = a$, and $\dot{x} = -\dot{y}$. Also $x^m \times y^n$ is a maximum; hence $mx^{m-1}\dot{x} \times y^n + ny^{n-1}\dot{y} \times x^m = 0$. Divide both sides by $x^{m-1}y^{n-1}$, then $my\dot{x} + nx\dot{y} = 0$; for $\dot{x}$ write $-\dot{y}$; $\therefore\ nx\dot{y} = my\dot{y}$, and $x : y :: m : n$.

Cor. If it be required to divide the given line into three parts $x$, $y$, and $z$, so that $x^m \times y^n \times z^p$ may be a maximum; it will follow by the reasoning in the last Cor. that $x : y :: m : n$, or

$y=\frac{nx}{m}$; and $x : z :: m : p$; $\therefore z=\frac{px}{m}$; hence, since $x+y+z=a$, we have $x+\frac{nx}{m}+\frac{px}{m}=a$, and $x=\frac{ma}{m+n+p}$. We may proceed in the same manner, whatever be the number of unknown quantities.

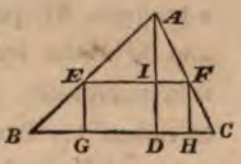
Ex. 4. To divide a given line *AB* into two parts, $x$ and $y$, so that $\frac{x}{y}+\frac{y}{x}$ may be a minimum.

As before, $x+y=a$, and $\dot{x}=-\dot{y}$; also $\frac{x}{y}+\frac{y}{x}$ is a minimum; $\therefore \frac{\dot{x}y-\dot{y}x}{y^2}+\frac{\dot{y}x-\dot{x}y}{x^2}=0$, or $-\frac{y\dot{y}+x\dot{y}}{y^2}+\frac{x\dot{y}+y\dot{y}}{x^2}=0$, by writing $-\dot{y}$ for $\dot{x}$; that is, $\frac{x\dot{y}+y\dot{y}}{y^2}=\frac{y\dot{y}+x\dot{y}}{x^2}$; $\therefore \frac{1}{y^2}=\frac{1}{x^2}$ and $x=y$, or the line must be bisected.

Ex. 5. To find the fraction, which shall exceed its cube by the greatest quantity possible.

Let the required fraction $=x$; then $x-x^3$ is a maximum; $\therefore \dot{x}-3x^2\dot{x}=0$, and $x=\frac{1}{\sqrt{3}}$.

Ex. 6. To inscribe the greatest rectangle in a given triangle, *ABC*.

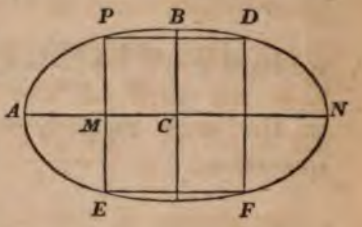


Draw *AD* perpendicular to *BC*, take $AD=a$, $BC=b$, $AI=x$, $EF=y$; then, by similar triangles,

$$a : b :: x : y=\frac{bx}{a};$$

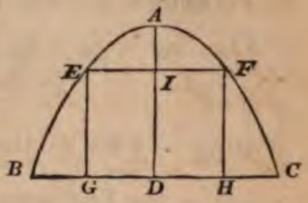
$\therefore$ the rectangle $EGHF=\frac{bx}{a}\times\overline{a-x}$ is a maximum; $\therefore ax-x^2$ is a maximum; $\therefore x=\frac{a}{2}$.

Ex. 7. To inscribe the greatest rectangle in a given ellipse *ABN*.



Let $CM = x$; $PM$ the ordinate $= y$; then, if $AC$ the semi-major axis $= a$, and $BC$ the semi-minor $= b$, the rectangle $PEFD = 2 \times \frac{b}{a} \times \sqrt{a^2 - x^2} \times 2x$; $\therefore x \times \sqrt{a^2 - x^2}$, or its square $a^2x^2 - x^4$, is a maximum; $\therefore$ the fluxion $= 0$, and $x = \frac{a}{\sqrt{2}}$.

Ex. 8. To inscribe the greatest rectangle in a given parabola *ABC*.

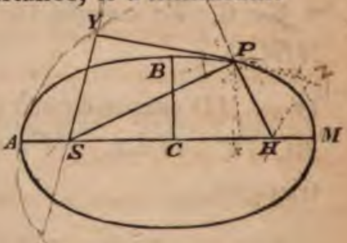


Draw the axis $AD$; let $AI = x$, $IF = y$, $AD = b$; latus rectum $= c$; then $FI = \sqrt{cx}$; $\therefore$ the rectangle $EH = 2\sqrt{cx} \times \overline{b - x}$; $\therefore bx^{\frac{1}{2}} - x^{\frac{3}{2}}$ is a maximum; its fluxion $= 0$, and $x = \frac{b}{3}$.

Ex. 9. Given the radius of a circle to determine the arc, when the rectangle under the sine and cosine is a maximum.

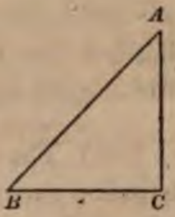
Let the radius $= r$, and the cosine $= x$; then the sine $= \sqrt{r^2 - x^2}$; $\therefore$ by the problem $x \times \sqrt{r^2 - x^2}$ is a maximum; hence the fluxion of $x^2 \times \overline{r^2 - x^2} = 0$, and $x = \frac{r}{\sqrt{2}}$.

Ex. 10. To determine in an ellipse, at what point the angle contained, between the tangent and distance, is a minimum.



Let $S$ and $H$ be the foci of the ellipse, $P$ the point required, $PY$ a tangent, $SY$ perpendicular to $PY$, and $SPY$ the least angle. Its sine is a minimum. Let $x =$ the sine to radius 1; then $1 : x :: SP : SY$; $\therefore x = \frac{SY}{SP} = \frac{1}{SP} \times BC \times \sqrt{\frac{SP}{HP}} \propto \frac{1}{\sqrt{SP \times HP}}$ a minimum; $\therefore SP \times HP$ is a maximum; and since $SP + HP = AM$, in this case $SP = HP$; and the point $P$ is at the extremity of the minor axis.

Ex. 11. Of all right-angled plane triangles having the same given hypothenuse, to find that whose area is greatest.



Let $AB=a$, $BC=x$; then $AC=\sqrt{a^2-x^2}$. Now the area of the triangle $ABC=\frac{BC\times AC}{2}$; $\therefore$ $\frac{x\times\sqrt{a^2-x^2}}{2}$, or $x^2\times\overline{a^2-x^2}$ is a maximum; $\therefore$ the fluxion of $a^2x^2-x^4$, or $2a^2x\dot{x}-4x^3\dot{x}=0$; hence $a^2=2x^2$, and $x=\frac{a}{\sqrt{2}}$.

Ex. 12. Of all right-angled triangles having the same area, to find that in which the sum of the sides $BC$, $CA$ is a minimum.

Let the area $=a$, $BC=x$; then, since $BC\times AC=2a$, $AC=\frac{2a}{x}$; $\therefore$ $x+\frac{2a}{x}$ is a minimum; hence $\dot{x}-\frac{2a\dot{x}}{x^2}=0$, and $x=\sqrt{2a}$.

Ex. 13. To find the value of $x$ and $y$ in the equation $\overline{x^2+y^2}|^2=a^2x^2$, when $y$ is a maximum.

Extract the square root; then $x^2+y^2=ax$; $\therefore$ $2x\dot{x}+2y\dot{y}=a\dot{x}$; but $\dot{y}$, and consequently $2y\dot{y}=0$, when $y$ is a maximum; $\therefore$ $2x\dot{x}=a\dot{x}$, and $x=\frac{a}{2}$. To find $y$, we have $\frac{a^2}{4}+y^2=\frac{a^2}{2}$; $\therefore$ $y^2=\frac{a^2}{4}$, and $y=\frac{a}{2}$.

(24.) To determine when the equation $x^3-9x^2+24x-16=0$ becomes a maximum or minimum.

Assume the fluxion $=0$; then $3x^2\dot{x}-18x\dot{x}+24\dot{x}=0$; or $3\dot{x}\times\overline{x^2-6x+8}=0$.

Now the roots of this quadratic equation are 2 and 4; $\therefore$ $3\dot{x}\times\overline{x-2}\times\overline{x-4}=0$.

To ascertain which of these roots gives a maximum, and which a minimum; find whether the value of the fluxion just before it $=0$, be positive or negative. If it be positive, the quantity is increasing, and the next root gives a maximum; if negative, it is decreasing, and the next root gives a minimum.

In this instance, if $\dot{x}$ be assumed positive, and $x$ less than 2, the value of $3\dot{x} \times \overline{x-2} \times \overline{x-4}$ is positive; $\therefore$ this root gives $x^3 - 9x^2 + 24x - 16$ a maximum. If $x$ be assumed greater than 2, but less than 4, $3\dot{x} \times \overline{x-2} \times \overline{x-4}$ is negative; $\therefore$ this root gives the original equation a minimum.

(25.) The meaning of the assertion, that if $x=2$ it gives the equation a maximum, and if it $=4$ a minimum, is, that $x^3 - 9x^2 + 24x - 16$ increases till $x=2$, and then decreases till $x=4$; not that it is the greatest possible when $x=2$, nor the least possible when $x=4$. For if quantities less than 2 be successively substituted for $x$, as

$$\left.\begin{array}{l} 1 \\ 0 \\ -1 \text{ \&c.} \end{array}\right\} \begin{array}{c} \text{- - - - - -} \\ \text{the results are} \\ \text{- - - - - -} \end{array} \left\{\begin{array}{r} 1 - 9 + 24 - 16, \text{ or } 0, \\ -16, \\ -1 - 9 - 24 - 16, \text{ or} -50 \text{ \&c.} \end{array}\right.$$

that is, it will go on decreasing, *sine limite.*

And if quantities greater than 4 be substituted successively for $x$, as

$$\left.\begin{array}{l} 5 \\ 6 \\ 7 \text{ \&c.} \end{array}\right\} \begin{array}{c} \text{- - - - - -} \\ \text{the results are} \\ \text{- - - - - -} \end{array} \left\{\begin{array}{r} 125 - 225 + 120 - 16, \text{ or } 4, \\ 216 - 324 + 144 - 16, \text{ or } 20, \\ 343 - 441 + 168 - 16, \text{ or } 54; \end{array}\right.$$

that is, it will go on increasing, *sine limite.*

(26.) In this case we have supposed $x$ to increase, and therefore that $\dot{x}$ is positive. If $x$ be a decreasing quantity, its fluxion is negative. Suppose $x$ to decrease till it becomes equal to 4; here $3\dot{x} \times \overline{x-2} \,.\, \overline{x-4}$ is negative, while $x$ is greater than 4; therefore, when $x=4$, the original quantity $x^3 - 9x^2 + 24x - 16$ is a minimum. If $x$ be assumed greater than 2, and less than 4, then $3\dot{x} \times \overline{x-2} \,.\, \overline{x-4}$ is positive; therefore the root 2 gives $x^3 - 9x^2 + 24x - 16$ a maximum. These results are exactly the same with those obtained by the first method.

(27.) When two or an even number of the roots of the resulting equation are equal, they shew neither a maximum nor a minimum.

It follows from the preceding articles, that when the fluxion of the given quantity is of the same denomination with regard to positive and negative, before and after it becomes equal to nothing, it does not indicate either a maximum or minimum. Now this occurs, when two roots of the fluxional equation are equal. For, let the given quantity be $3x^4-32x^3+120x^2-192x$; of which the fluxion is $12x^3\dot{x}-96x^2\dot{x}+240x\dot{x}-192\dot{x}$;

$$\text{or, } 12\dot{x}\times\overline{x^3-8x^2+20x-16};$$
$$\text{or, } 12\dot{x}\times\overline{x-2}\times\overline{x-2}\times\overline{x-4}.$$

Let $\dot{x}$ be positive; then before $x=2$ this fluxion is negative; and if $x$ be greater than 2, and less than 4, it is still negative; therefore the root 2 does not give a minimum. But as the fluxion changes from $-$ to $+$, while $x$ increases from a quantity less than 4, to a quantity greater than 4, this root 4 gives $3x^4-32x^3+120x^2-192x$ a minimum; and it then begins to increase.

In the same manner, if the fluxional equation has 4 equal roots, as $\dot{x}\times\overline{x-a}\times\overline{x-a}\times\overline{x-a}\times\overline{x-a}\times\overline{x-2a}$, or any even number, the fluxion is of the same denomination with respect to $+$ and $-$, both before and after $x$ becomes equal to $a$; and therefore the equal roots neither indicate a maximum nor a minimum.

(28.) The number of maxima or minima which a flowing quantity admits, is equal to the number of unequal roots in the fluxional equation.

Let $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4=0$ be an equation, in which it is required to determine the different values of $x$, when the expression becomes a maximum or minimum. Put the fluxion $=0$;

$$\therefore\ 12x^3\dot{x}-84ax^2\dot{x}+168a^2x\dot{x}-96a^3\dot{x}=0;$$
$$\text{or, } 12\dot{x}\times\overline{x^3-7ax^2+14a^2x-8a^3}=0;$$
$$\text{or, } 12\dot{x}\times\overline{x-a}\times\overline{x-2a}\times\overline{x-4a}.$$

If $x$ be assumed less than $a$, the result is $-$, or the root $a$ indicates a minimum; if $x$ be greater than $a$, but less than $2a$, the result is $+$; and the root $2a$ denotes a maximum, &c.; therefore when all the roots are unequal, the proposition is true.

And if the fluxional equation have an odd number of equal roots, as $\dot{x}\times\overline{x-a}\times\overline{x-a}\times\overline{x-a}\times\overline{x-2a}$, when $x$ is less than $a$, the result is + ; when greater than $a$, but less than $2a$, it is − ; therefore one root $a$ gives a maximum, and $2a$ a minimum; the product of $\overline{x-a} \times \overline{x-a}$ determines nothing; hence universally, there are as many maxima and minima, as unequal roots, in the given equation.

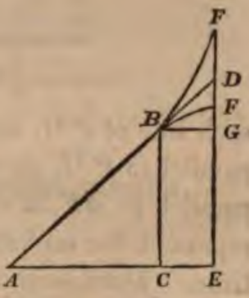
When all the roots are impossible in the fluxional equation, as no possible value of $x$ can give a result $= 0$, the quantity must either increase or decrease perpetually, and therefore cannot admit a maximum or minimum.

(29.) Every quantity which admits a maximum or minimum is of a compound nature; one part of it must increase, while another decreases, and according to the increase or decrease it approaches a maximum or a minimum. Thus, in Ex. 1. Art. 23, we have two quantities, $2ax$ and $-x^2$; if $x$ increases, $2ax$ also increases; but $x^2$ increases at the same time; therefore the expression $2ax-x^2$ partly increases, and partly diminishes; this quantity, then, is in a state to admit a maximum or a minimum.

## CHAP. III.

### THE METHOD OF DRAWING TANGENTS TO CURVES.

(30.) IF a straight line as $BC$ move on the line $AC$ in a direction always parallel to itself, and $AC$ and $CB$ increase uniformly, the locus of the point $B$ will be a straight line; and the motion of $B$ in that straight line will be uniform.



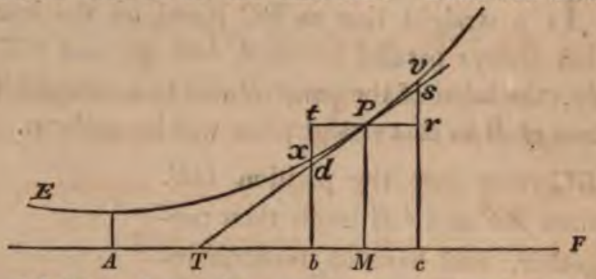
Let $BC$ come into the position $DE$. Then, since $AC$ and $CB$ begin their motion together, and have an uniform increase, the ratio of $AC : CB$ is constant; that is, $AC : CB :: AE : ED$, or $ACB$ and $AED$ are similar triangles; $\therefore ABD$ is a straight line. Also the motion in that line is uniform; for, since $CB$ and $ED$ are parallel, $AC : CE :: AB : BD$, and alternately $AC : AB :: CE : BD$; but $AC$ is to $AB$ in a constant ratio; $\therefore CE : BD$ in a constant ratio; and the motion in the direction $AE$ is uniform; $\therefore$ the motion in the direction $AD$ is uniform.

COR. 1. If the motion in the direction $AC$ be uniform, but that in the direction $CB$ not uniform, the point $B$ will trace out a curve. The same construction remaining, let $CB$ increase with an accelerated velocity; then $BG$ being drawn parallel to $CE$, $BG$ and $GD$ would represent the uniform contemporaneous increments of $AC$, and $CB$; but if $CB$ increase with an accelerated velocity whilst the velocity of $C$ is uniform, $BG$ and some line $GF$ greater than $GD$ will represent the corresponding increments; in this case, a curve $BF$ is described convex to the line $AE$. By the same reasoning, if the increment of $CB$ is perpetually retarded, whilst that of $AC$ remains uniform, the point $B$ will trace out a curve, which is concave to the line $AE$.

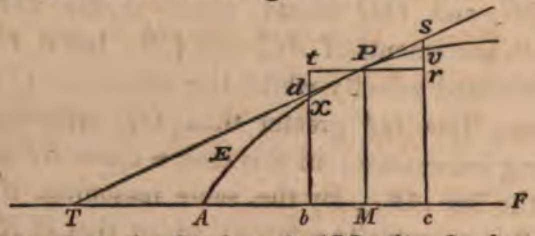
Cor. 2. If $AC$ increase with an uniform velocity, but the increase of $CB$ is not uniform, the curve $BF$ is not described with an uniform motion.

(31.) To draw a tangent to any algebraic curve.

Let $AM$ represent the abscissa, and $MP$ the ordinate of an algebraic curve convex to the axis $AF$. Take $Mc$ and $Mb$ on each side of $PM$, and equal to each other; and draw $bt$ and $cv$ parallel to $PM$, meeting the curve in $x$ and $v$, and a line $tPr$ parallel to $AF$ in $t$ and $r$. Let $Mb$, and $Mc$, or $Pt$ and $Pr$ represent the uniform increase of the abscissa $AM$ in a given time. Then since $EPV$ is convex to the axis, $MP$ increases with an accelerated velocity; $\therefore$ the fluxion of $AM$ : the fluxion of $MP$ :: $Pr$ : a quantity less than $rv$. Take $rs$ equal to this quantity; join $Ps$, and produce it both ways; this line is a tangent, that is, every part of it falls below the curve. For since by equal triangles $Ptd$, $Prs$, $rs=td$; $\therefore td$ is the fluxion of the ordinate at $P$. But the fluxion of $AM$ : the fluxion of $MP$ :: $Pr$ or $Pt$ : a quantity greater than the preceding increment $tx$; $\therefore td$ is greater than $tx$, and $d$ is below the curve.



Next, suppose the curve to be concave to the axis $AF$. The same construction remaining, since the increase of $MP$ is in this case retarded, the fluxion of $AM$ : the fluxion of $MP$ ::



$Pr$ : a quantity greater than $rv$; take $rs$ equal to this quantity; join $sP$, and produce it; this line is a tangent at $P$, or falls wholly above the curve. For, by equal and similar triangles, $Prs$, $Ptd$, $td=rs$, or $=$ the fluxion of the ordinate at $P$.

But the fluxion of $AM$ : the fluxion of $MP$ :: $Pt$ or $Pr$ : a quantity less than $tx$; $\therefore$ $td$ is less than $tx$, or $d$ is above the curve.

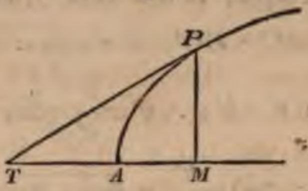
In both these cases, if $AM=x$, $PM=y$; $Mc=\dot{x}$, $rs=\dot{y}$, we have $rs$ : $rP$ or $Mc$ :: $PM$ : $MT$ by similar triangles;

or $\dot{y} : \dot{x} :: y$ : the subtangent $MT=\dfrac{y\dot{x}}{\dot{y}}$.

Hence in any algebraic curve, to which it is required to draw a tangent at any point $P$, find from the equation to the curve the value of $\dfrac{y\dot{x}}{\dot{y}}$; take $MT$ equal to this expression, join $TP$, and produce it. $TP$ is the tangent required.

## EXAMPLES.

(32.) Ex. 1. To draw a tangent to the common parabola.



Let $AP$ be the parabola, $AM$ the axis, and $P$ the point at which the tangent is to be drawn. Take $AM=x$, $PM=y$, the principal latus rectum $=a$; then $y^2=ax$;

$$\therefore\ 2y\dot{y}=a\dot{x};\ \therefore\ \frac{\dot{x}}{\dot{y}}=\frac{2y}{a},\text{ and }\frac{y\dot{x}}{\dot{y}}=\frac{2y^2}{a}=\frac{2ax}{a}=2x;$$

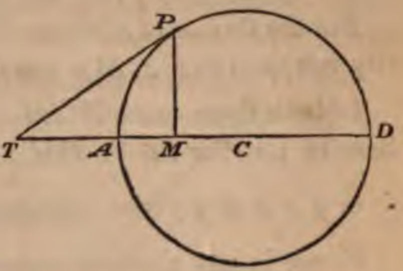
that is, $MT$ the subtangent $=$ twice the abscissa $MA$. Hence, to draw a tangent at $P$, let fall the ordinate $PM$, and in $AM$ produced, take $MT=2MA$, and join $TP$. $TP$ is the tangent.

Ex. 2. In general, to draw a tangent to any parabola of which the equation is $a^{n-1}x=y^n$.

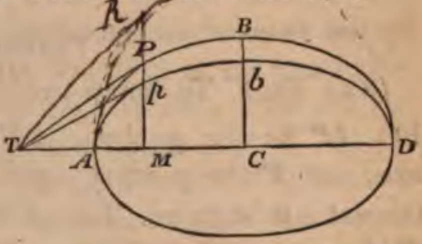
Here $a^{n-1}\dot{x}=ny^{n-1}\dot{y}$; $\therefore\ \dfrac{\dot{x}}{\dot{y}}=\dfrac{ny^{n-1}}{a^{n-1}}$; $\therefore$ $MT$ or $\dfrac{y\dot{x}}{\dot{y}}=$ $\dfrac{ny^n}{a^{n-1}}=\dfrac{na^{n-1}x}{a^{n-1}}=nx$ the subtangent.

Cor. If $n=2$ it becomes the common parabola, and $MT=2x$ as before.

Ex. 3. To draw a tangent to the circle at a given point $P$. Let fall the ordinate $PM$ on the diameter $AD$. Let $AM=x$; $PM=y$; $AD=2a$; then $y^2=2ax-x^2$; $\therefore 2y\dot{y}=2a\dot{x}-2x\dot{x}$; $\therefore \frac{\dot{x}}{\dot{y}}=\frac{y}{a-x}$, $\therefore \frac{y\dot{x}}{\dot{y}}$ or $MT=\frac{y^2}{a-x}=\frac{2ax-x^2}{a-x}$. Take $MT=$ to this quantity, and join $TP$; $TP$ is the tangent.



Ex. 4. To draw a tangent to the ellipse $ABD$ at any point $P$. Let $AC=a$, $CB=b$, where $AC$ and $CB$ are the semi-axes; $AM=x$, $PM=y$; then, by the property of the ellipse, $AM\times MD : MP^2 :: AC^2 : BC^2$, or $x\times\overline{2a-x} : y^2 :: a^2 : b^2$; $\therefore y^2=\frac{b^2}{a^2}\times\overline{2ax-x^2}$;



$$\therefore 2y\dot{y}=\frac{b^2}{a^2}\times\overline{2a\dot{x}-2x\dot{x}}=\frac{b^2}{a^2}\times\overline{a-x}\times 2\dot{x},\ \therefore\frac{\dot{x}}{\dot{y}}=\frac{y}{\frac{b^2}{a^2}\times\overline{a-x}};$$

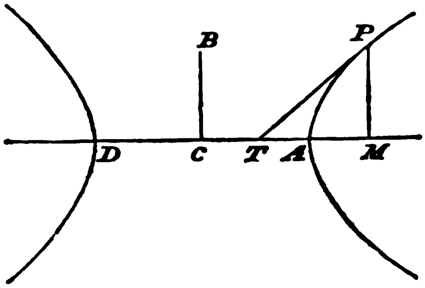
$$\therefore MT \text{ or } \frac{y\dot{x}}{\dot{y}}=\frac{y^2}{\frac{b^2}{a^2}\times\overline{a-x}}=\frac{\frac{b^2}{a^2}\times\overline{2ax-x^2}}{\frac{b^2}{a^2}\times\overline{a-x}}=\frac{2ax-x^2}{a-x}.$$

Cor. 1. Since the value of $MT$ is expressed in terms of $a$ and $x$, the subtangent will continue the same, if $a$ and $x$ are the same, whatever be the minor axis of the ellipse; therefore if any other ellipse $ApD$ be described on the same major axis, and cut $MP$ in $p$, and $Tp$ be joined, $Tp$ is also a tangent to this ellipse at the point $p$. And if a circle, described upon the major axis, as a diameter, cut $MP$ produced in $R$, and $TR$ be joined, $TR$ is a tangent to the circle.

**Cor. 2.** If $x$ be greater than $a$, and less than $2a$, $\frac{2ax - x^2}{a - x}$ is negative; or the sub-tangent lies the contrary way.

Ex. 5. To draw a tangent to the hyperbola $AP$, whose major axis is $AD$, and minor $2BC$.

Let $AC = a$; $CB = b$; $AM = x$; $MP = y$.



By the nature of the hyperbola,

$$AM \times \underline{MD} : MP^2 :: AC^2 : CB^2;$$

$$\text{or,}\quad x \times \overline{2a+x} : y^2 :: a^2 : b^2;$$

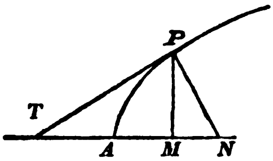
$$\therefore y^2 = \frac{b^2}{a^2} \times \overline{2ax+x^2};$$

$$\therefore 2y\dot{y} = \frac{b^2}{a^2} \times \overline{2a\dot{x}+2x\dot{x}} = \frac{b^2}{a^2} \times \overline{a+x} \times 2\dot{x};$$

$$\therefore \frac{\dot{x}}{\dot{y}} = \frac{y}{\frac{b^2}{a^2} \times \overline{a+x}}; \therefore MT \text{ or } \frac{y\dot{x}}{\dot{y}} = \frac{y^2}{\frac{b^2}{a^2} \times \overline{a+x}} = \frac{\frac{b^2}{a^2} \times \overline{2ax+x^2}}{\frac{b^2}{a^2} \times \overline{a+x}}$$

$$= \frac{2ax+x^2}{a+x}.$$

(33.) **Def.** If $PN$ be drawn perpendicular to $PT$, meeting the axis in $N$, $PN$ is called the normal, and $MN$ the sub-normal.



To determine their values, we have $TM : MP :: MP : MN$; or $\frac{y\dot{x}}{\dot{y}} : y :: y : MN$ the sub-normal $= \frac{y\dot{y}}{\dot{x}}$.

Also, $NP^2 = NM^2 + MP^2 = \frac{y^2\dot{y}^2}{\dot{x}^2} + y^2 = \frac{y^2 \times \overline{\dot{x}^2+\dot{y}^2}}{\dot{x}^2}$;

∴ $NP$ the normal $= \frac{y \times \sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$.

## (34.) EXAMPLES.

Ex. 1. To find the value of the normal and sub-normal in the common parabola.

Here $y^2 = ax$; ∴ $2y\dot{y} = a\dot{x}$, and $\frac{y\dot{y}}{x} = \frac{a}{2}$ the sub-normal;

Also, $\dot{x} = \frac{2y\dot{y}}{a}$; ∴ $y^2 \times \frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2} = y^2 \times \overline{1+\frac{a^2\dot{y}^2}{4y^2\dot{y}^2}} = y^2 + \frac{a^2}{4}$;
∴ the normal $PN = \sqrt{y^2+\frac{1}{4}a^2}$.

Ex. 2. To find the value of the sub-normal in the ellipse and hyperbola.

In the ellipse, $y^2 = \frac{b^2}{a^2} \times \overline{2ax - x^2}$;

$$\therefore\ y\dot{y} = \frac{b^2}{a^2} \times \overline{a\dot{x} - x\dot{x}};$$

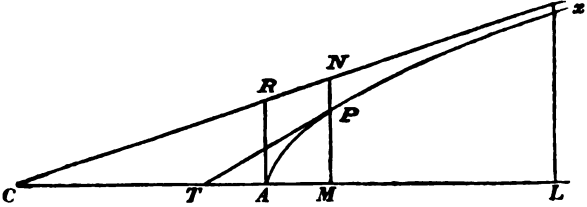
$$\text{and } \frac{y\dot{y}}{\dot{x}} = \frac{b^2}{a^2} \times \overline{a - x} = \text{the sub-normal.}$$

In the hyperbola, it $= \frac{b^2}{a^2} \times \overline{a+x}$.

# CHAP. IV.

## TO DRAW ASYMPTOTES TO CURVES.

(35.) DEF. AN asymptote to a curve is a straight line cutting the axis at a finite distance, which continually approaches the curve, and arrives nearer than by any assignable difference, but never meets it, though indefinitely produced.



Let $AP$ represent a curve, which admits an asymptote $Cx$; this line is conceived to be a tangent to the curve, or the limit to which the tangent approaches at an infinite distance. Take $AM = x$, $MP = y$; let $MP$ produced meet the asymptote in $N$, and draw $AR$ perpendicular to $AM$, and $PT$ a tangent at $P$. From the nature of the curve proposed, find the value of $\frac{y\dot{x}}{\dot{y}} = MT$ the sub-tangent; hence $AT$ may be found $= \frac{y\dot{x}}{\dot{y}} - x$. Imagine $x$ to become infinite, and $T$ to move on to $C$; if $AC$ be finite, the curve admits an asymptote. Next, find the ratio of $TM : MP$ when $x$ is infinite; that is, $Lx$ being supposed an ordinate at an infinite distance, the ratio of $CL : Lx$; then, by similar triangles, $CLx$, $CAR$, $CL : Lx :: CA : AR$; of which proportion the three first terms are known, and therefore $AR$ can be determined. Join $CR$, and produce it indefinitely. $CR$ is the asymptote required.

(36.) EXAMPLES.

Ex. 1. To draw an asymptote to the common hyperbola. (See the preceding FIGURE.)

By Art. 32. Ex. 5. $MT = \frac{2ax + x^2}{a + x}$; $\therefore AT = \frac{2ax + x^2}{a + x} - x$

$= \frac{ax}{a+x}$. Let $x$ become infinite; $AT$ ultimately $= \frac{ax}{x} = a =$ $AC$. Now if $x$ is infinite, $TM$, which then becomes $CL$, $= \frac{x^2}{x} = x$; and $MP$, or $\frac{b}{a} \times \sqrt{2ax+x^2}$, which in that case $= Lx$ $= \frac{bx}{a}$; $\therefore$ since $CL : Lx :: CA : AR$,

$$x : \frac{bx}{a} :: a : AR; \quad \therefore AR = b.$$

Hence, from $A$ draw $AR$ perpendicular to $CA$, and equal to $b$. Take the centre $C$; join $CR$, and produce it indefinitely. $CRx$ is the asymptote.

Ex. 2. Let the equation to the curve be $y^3 = ax^2 + x^3$.

Here $3y^2\dot{y} = 2ax\dot{x} + 3x^2\dot{x}$; $\therefore \frac{y\dot{x}}{\dot{y}} = \frac{3y^3}{2ax+3x^2} = \frac{3ax^2+3x^3}{2ax+3x^2}$ $= MT$. Hence $AT = \frac{3ax^2+3x^3}{2ax+3x^2} - x = \frac{ax^2}{2ax+3x^2}$. Let $x$ become infinite, $CL = \frac{3x^3}{3x^2} = x$; $Lx = \sqrt[3]{ax^2+x^3} = x$; $AC$ $= \frac{ax^2}{3x^2} = \frac{a}{3}$. Therefore, since $CL : Lx :: CA : AR$; $x : x :: \frac{a}{3}$ $: AR = \frac{a}{3}$.

Hence, draw $AR$ perpendicular to $AC$, and $= AC = \frac{a}{3}$, and join $CR$. $CR$ produced indefinitely will be the asymptote.

## CHAP. V.

### ON THE METHOD OF FINDING FLUENTS.

(37.) By the *direct* method of fluxions, we are taught how to find a fluxion from a fluent. The object of the *inverse* method, which is deduced from the former, is to find the fluent from the fluxion. In the former case, general rules are proposed, which are easy in the application; but it is frequently difficult to determine the fluent of a given fluxion, and in some cases even impossible; for it is obvious, that certain fluxions may be of such a nature as could not result from taking the fluxion of any fluent whatever. Rules can only be proposed for finding the fluents of those fluxions, whose forms prove them to have been deduced from some fluents.

(38.) To find the fluent of any power of a simple quantity which is multiplied by the fluxion of that quantity.

The fluxion of $x^4$ is $4x^3\dot{x}$; $\therefore$ the fluent of $4x^3\dot{x}$ is $x^4$.

The fluxion of $x^6$ is $6x^5\dot{x}$; $\therefore$ the fluent of $6x^5\dot{x}$ is $x^6$.

And, in general, the fluxion of $x^n$ is $nx^{n-1}\dot{x}$; $\therefore$ the fluent of $nx^{n-1}\dot{x}$ is $x^n$.

Hence, to find the fluent, we have the following

RULE. Divide by the fluxion of the root, add 1 to the index, and divide by the index thus increased.

EXAMPLES.

Ex. 1. The fluent of $10x^9\dot{x} = \dfrac{10x^{9+1}\dot{x}}{10\dot{x}} = x^{10}$.

Ex. 2. The fluent of $40x^4\dot{x} = 8x^5$.

Ex. 3. The fluent of $3x^7\dot{x} = \dfrac{3x^8}{8}$.

Ex. 4. The fluent of $5x^{\frac{5}{3}}\dot{x} = \dfrac{5x^{\frac{8}{3}}}{\frac{8}{3}} = \dfrac{15x^{\frac{8}{3}}}{8}$.

Ex. 5. The fluent of $\frac{4\dot{x}}{3x^{\frac{1}{2}}}$ or $\frac{4\dot{x} \times x^{-\frac{1}{2}}}{3} = \frac{4x^{\frac{1}{2}}}{\frac{1}{2} \times 3} = \frac{8x^{\frac{1}{2}}}{3}$.

Ex. 6. The fluent of $\frac{3x^2\dot{x}}{2x^{\frac{1}{2}}}$ or of $\frac{3x^{\frac{3}{2}}\dot{x}}{2} = \frac{3x^{\frac{5}{2}}}{2 \times \frac{5}{2}} = x^{\frac{5}{2}}$.

(39.) If the quantity be compound, its fluent in certain cases may be found by the same Rule.

The fluxion of $\overline{a^2+x^2}|^3 = 6x\dot{x} \times \overline{a^2+x^2}|^2$; $\therefore$ the fluent of $6x\dot{x} \times \overline{a^2+x^2}|^2$ is $\overline{a^2+x^2}|^3$; and it is obtained by the Rule preceding, in the following manner; $a^2+x^2$ is called the root; its fluxion is $2x\dot{x}$; divide by the fluxion of the root, and the result is $3 \times \overline{a^2+x^2}|^2$; add 1 to the index, and divide by it; the answer is $\overline{a^2+x^2}|^3$; $\therefore$ in this case the Rule gives the correct fluent. And, in general, since the fluxion of $\overline{a^n+x^n}|^p = pnx^{n-1}\dot{x} \times \overline{a^n+x^n}|^{p-1}$, and that the Rule already laid down will from this fluxion deduce the right fluent $\overline{a^n+x^n}|^p$, it follows that the Rule holds in all fluxions of a similar nature; that is, wherever the index of the variable quantity without the vinculum is less by 1 than the index under the vinculum.

### EXAMPLES.

Ex. 1. The fluent of $\overline{a^2+x^2}|^3 \times 6x\dot{x} = \frac{\overline{a^2+x^2}|^4 \times 6x\dot{x}}{2x\dot{x} \times 4} = \overline{a^2+x^2}|^4 \times \frac{6}{8} = \frac{3}{4} \times \overline{a^2+x^2}|^4$.

Ex. 2. The fluent of $\overline{a^5-x^5}|^{\frac{3}{4}} \times x^4\dot{x} = \frac{\overline{a^5-x^5}|^{\frac{7}{4}} \times x^4\dot{x}}{-5x^4\dot{x} \times \frac{7}{4}} = -\frac{4}{35} \times \overline{a^5-x^5}|^{\frac{7}{4}}$.

Ex. 3. The fluent of $\frac{4x^3\dot{x}}{\overline{a^4+x^4}|^{\frac{1}{2}}}$, or $\overline{a^4+x^4}|^{-\frac{1}{2}} \times 4x^3\dot{x} = \frac{\overline{a^4+x^4}|^{\frac{1}{2}} \times 4x^3\dot{x}}{4x^3\dot{x} \times \frac{1}{2}} = 2 \times \overline{a^4+x^4}|^{\frac{1}{2}}$.

Ex. 4. The fluent of $\overline{a^6+4x^6}|^{\frac{3}{2}} \times x^5\dot{x} = \frac{\overline{a^6+4x^6}|^{\frac{5}{2}} \times x^5\dot{x}}{24x^5\dot{x} \times \frac{5}{2}} = \frac{\overline{a^6+4x^6}|^{\frac{5}{2}}}{60}$.

Ex. 5. To find the fluent of $\overline{a+x}|^4 \times \dot{x}$. Since $x^0 = 1$, this is the same as $\overline{a+x}|^4 \times x^0\dot{x}$; and the index of $x$ without the vinculum is one less than the index of $x$ under the vinculum; therefore the Rule applies, and the fluent $=$ $\frac{\overline{a+x}|^5 \times \dot{x}}{\dot{x} \times 5} = \frac{\overline{a+x}|^5}{5}$. In the same manner,

Ex. 6. The fluent of $\overline{9y+4a}|^{\frac{1}{2}} \times \dot{y} = \frac{2}{27} \times \overline{9y+4a}|^{\frac{3}{2}}$.

(40.) If all the quantities under the vinculum be variable, and the quantity without be in any given ratio to the fluxion of the root, the fluent may be found as before.

Ex. 1. Let $\overline{x^2+y^2}|^3 \times \overline{6x\dot{x}+6y\dot{y}}$ be the fluxion, whose fluent is required; the root is $x^2+y^2$, and its fluxion $= 2x\dot{x}+2y\dot{y}$; $\therefore$ the fluent $= \frac{\overline{x^2+y^2}|^4 \times \overline{6x\dot{x}+6y\dot{y}}}{\overline{2x\dot{x}+2y\dot{y}} \times 4} = \frac{3}{4} \times \overline{x^2+y^2}|^4$.

Ex. 2. The fluent of $\overline{x^4+y^6+z^8}|^{\frac{1}{3}} \times \overline{8x^3\dot{x}+12y^5\dot{y}+16z^7\dot{z}} =$ $\frac{\overline{x^4+y^6+z^8}|^{\frac{4}{3}} \times \overline{8x^3\dot{x}+12y^5\dot{y}+16z^7\dot{z}}}{\overline{4x^3\dot{x}+6y^5\dot{y}+8z^7\dot{z}} : \times \frac{3}{4}} = \frac{\overline{x^4+y^6+z^8}|^{\frac{4}{3}} \times 4}{3}$.

Ex. 3. The fluent of $\overline{a^2x^2+x^4}|^{\frac{1}{2}} \times \overline{2a^2x\dot{x}+4x^3\dot{x}} = \frac{2}{3} \times \overline{a^2x^2+x^4}|^{\frac{3}{2}}$.

## TO FIND FLUENTS BY LOGARITHMS.

(41.) Let $y$ be any number, and $x$ its logarithm; then if $x$ increase uniformly, or in arithmetic progression, by the nature of logarithms $y$ increases in geometric; but if quantities increase in geometric progression, their differences or increments are proportional to the quantities themselves; that is, ultimately, $y \propto \dot{y}$, and $\frac{\dot{y}}{y}$ is constant; but $\dot{x}$ is constant, $\therefore \frac{\dot{y}}{y} \propto \dot{x}$; and if $m$ be assumed of a proper magnitude, $\dot{x} = \frac{m\dot{y}}{y}$; or the fluxion of any logarithm is equal to some constant quantity multiplied into the fluxion of the number, and this product divided by the number itself.

This quantity $m$ is called the modulus of the system.

If $m = 1$, $\dot{x} = \frac{\dot{y}}{y}$, an equation which may be deduced from the hyperbola; hence these logarithms are called hyperbolic.

Cor. If $x =$ the hyperbolic logarithm of $y$, $\dot{x} = \frac{\dot{y}}{y}$; and conversely, if $\dot{x} = \frac{\dot{y}}{y}$, $x =$ hyp. log. of $y$. Now $y$ may represent any compound number, and $\dot{y}$ the fluxion of that number; hence, if any fluxional expression consist of the fluxion of a quantity divided by that quantity itself, the fluent will be the hyperbolic logarithm of the quantity.

(42.) Examples.

Ex. 1. The fluent of $\frac{\dot{x}}{1+x}$ = hyp. log. $1+x$.

Ex. 2. The fluent of $\frac{2x\dot{x}}{a^2+x^2}$ = hyp. log. $a^2+x^2$.

Ex. 3. The fluent of $\frac{bx^2\dot{x}}{a^3+x^3} = \frac{b}{3} \times$ fluent of $\frac{3x^2\dot{x}}{a^3+x^3} = \frac{b}{3} \times$ hyp. log. $a^3+x^3$.

Ex. 4. The fluent of $\frac{x^{n-1}\dot{x}}{a^n+x^n} = \frac{1}{n} \times$ fluent of $\frac{nx^{n-1}\dot{x}}{a^n+x^n} = \frac{1}{n} \times$ hyp. log. $a^n+x^n$.

Ex. 5. The fluent of $\frac{y\dot{y}}{1+\frac{4y^2}{a^2}}$ = fluent of $\frac{a^2}{8} \times \frac{2y\dot{y}}{\frac{a^2}{4}+y^2} = \frac{a^2}{8} \times$ hyp. log. $\frac{a^2}{4} + y^2$.

(43.) In these and similar cases, the application of the Rule is obvious; but as these instances do not often occur, it is generally necessary, in fluxional expressions which admit fluents by logarithms, to use one of the following forms.

**Form I.** The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ = hyp. log. $\overline{x+\sqrt{x^2 \pm a^2}}$.

Let $\sqrt{x^2 \pm a^2} = v$, then $\overline{x^2 \pm a^2} = v^2$, and $x\dot{x} = v\dot{v}$;

$\therefore\ \dot{x} : \dot{v} :: v : x$,

and $\dot{x} : \dot{x}+\dot{v} :: v : v+x$; $\therefore\ \frac{\dot{x}}{v} = \frac{\dot{x}+\dot{v}}{x+v}$, or $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}+\dot{v}}{x+v}$; $\therefore$ the fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ = hyp. log. $\overline{x+v}$ = hyp. log. $\overline{x+\sqrt{x^2 \pm a^2}}$.

**Form II.** The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}}$ is the hyp. log. of $\overline{x \pm a + \sqrt{x^2 \pm 2ax}}$.

Let $\sqrt{x^2 \pm 2ax} = v$; then $x^2 \pm 2ax + a^2 = v^2 + a^2$, and $x \pm a = \sqrt{v^2 + a^2}$; $\therefore\ \dot{x} = \frac{v\dot{v}}{\sqrt{v^2+a^2}}$, and $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}} = \frac{\dot{v}}{\sqrt{v^2+a^2}}$; $\therefore$ the fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}}$ = fluent of $\frac{\dot{v}}{\sqrt{v^2+a^2}}$ = hyp. log. $\overline{v+\sqrt{v^2+a^2}}$ = hyp. log. of $\overline{x \pm a + \sqrt{x^2 \pm 2ax}}$.

**Form III.** The fluent of $\frac{2a\dot{x}}{a^2-x^2}$ is the hyp. log. of $\frac{a+x}{a-x}$.

Assume $\frac{A}{a+x} + \frac{B}{a-x} = \frac{2a}{a^2-x^2}$; then $\frac{Aa - Ax + Ba + Bx}{a^2-x^2} = \frac{2a}{a^2-x^2}$, or $\left\{\begin{array}{l} Aa+Bx \\ +Ba-Ax \\ -2a \end{array}\right\} = 0$; hence, equating the homologous terms, $Bx - Ax = 0$; $\therefore B = A$;

also, $Aa + Ba - 2a = 0$, or $2Aa = 2a$; $\therefore A = 1$, and $B = 1$;

$\therefore\ \frac{2a\dot{x}}{a^2-x^2} = \frac{\dot{x}}{a+x} + \frac{\dot{x}}{a-x} = \frac{\dot{x}}{a+x} - 1 \times \frac{-\dot{x}}{a-x}$, and the fluent = hyp. log. $a+x$ − hyp. log. $a-x$ = hyp. log. $\frac{a+x}{a-x}$.

In the same manner the fluent of $\frac{2a\dot{x}}{x^2-a^2}$ = hyp. log. $\frac{x-a}{x+a}$.

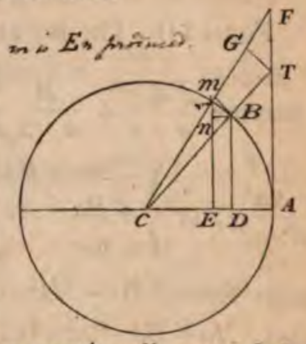
Form IV. The fluent of $\frac{2a\dot{x}}{x\sqrt{a^2+x^2}}$ is the hyp. log. of $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$.

Let $\sqrt{a^2+x^2}=v$; then $a^2+x^2=v^2$; and $x\dot{x}=v\dot{v}$; $\therefore\ \frac{2a\dot{x}}{xv}=\frac{2a\dot{v}}{x^2}$, or $\frac{2a\dot{x}}{x\sqrt{a^2+x^2}}=\frac{2a\dot{v}}{v^2-a^2}$; and the fluent of $\frac{2a\dot{x}}{x\sqrt{a^2+x^2}}=$ hyp. log. $\frac{v-a}{v+a}=$ hyp. log. of $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$.

In the same manner, the fluent of $\frac{2a\dot{x}}{x\sqrt{a^2-x^2}}=$ hyp. log. $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}$.

These are the principal forms of fluxions, whose fluents may be found by a table of hyperbolic logarithms. This table may be supplied by a table of the common form; for the hyberbolic logarithm of any number : the common log. :: 1 : $m$ the modulus of the common system. This subject is more fully explained under the Article Logarithms.

(44.) To find fluents by means of circular arcs.



Let $AB$ be a circular arc, whose center is $C$; $BD$ the right sine, $AT$ the tangent, $CT$ the secant; let $Bn$, $nm$ represent the fluxion of $AD$ and $BD$; then $Bm$ is the fluxion of the arc. For $Bn$ and $nm$ being described uniformly, $Bm$ is described uniformly, and is in the direction of a tangent at $B$; also, since the arc is described by a velocity either continually accelerated or continually retarded, $Bm$ is a limit between the increments. Conceive $Bn$ and $nm$ to be very small. Join $Cm$, and let it meet $AT$ produced in $F$; draw $TG$ perpendicular to $CF$. Assume $CA=a$, $AD=x$, $AB=z$, $DB=y$, $AT=t$, $CT=s$, $Bn=\dot{x}$, $nm=\dot{y}$, $Bm=\dot{z}$; and since ultimately the fluxion = the increment, $TF=\dot{t}$, and $GF=\dot{s}$.

Case I. To find the fluxion of a circular arc in terms of the right sine.

By the similar triangles *Bmn*, *CBD*,

$Bm : mn :: CB : CD$

or $\dot{z} : \dot{y} :: a : \sqrt{a^2-y^2}$; $\therefore \dot{z} = \dfrac{a\dot{y}}{\sqrt{a^2-y^2}}$.

Case II. To find the fluxion of a circular arc in terms of the versed sine.

By the same triangles,

$Bm : Bn :: CB : BD$

or $\dot{z} : \dot{x} :: a : \sqrt{2ax-x^2}$;

$$\therefore \dot{z} = \frac{a\dot{x}}{\sqrt{2ax-x^2}}.$$

Case III. To find the fluxion of a circular arc in terms of the tangent.

By the similar triangles *CBm*, *CTG*,

$Bm : TG :: CB : CT$. And by similar triangles *FTG*, *ACT*,

$TG : TF :: AC : CT$;

$\therefore Bm : TF :: AC \times CB : CT^2$; or

$\dot{z} : \dot{t} :: a^2 : a^2+t^2$;

$$\therefore \dot{z} = \frac{a^2\dot{t}}{a^2+t^2}.$$

Case IV. To find the fluxion of a circular arc in terms of the secant.

By the same triangles,

$Bm : TG :: CB : CT$

$TG : GF :: CA : AT$;

$\therefore Bm : GF :: AC \times BC : AT \times CT$; or

$\dot{z} : \dot{s} :: a^2 : \sqrt{s^2-a^2} \times s$;

$$\therefore \dot{z} = \frac{a^2\dot{s}}{s \times \sqrt{s^2-a^2}}.$$

These are the four principal forms. But it may be useful to add another.

CASE V. To find the fluxion of a circular arc in terms of the cosine.

In this case let $CD=x$, $Bn=-\dot{x}$; for as the arc increases $CD$ diminishes; and $BD=\sqrt{a^2-x^2}$.

Hence, by similar triangles,

$$Bm : Bn :: CB : BD;$$

$$\text{or } \dot{z} : -\dot{x} :: a : \sqrt{a^2-x^2};$$

$$\therefore \dot{z} = \frac{-a\dot{x}}{\sqrt{a^2-x^2}}.$$

COR. 1. The fluent of $\dfrac{a\dot{y}}{\sqrt{a^2-y^2}}=z=$ a circular arc, whose radius is $a$, and sine $y$.

COR. 2. The fluent of $\dfrac{a\dot{x}}{\sqrt{2ax-x^2}}$ is a circular arc, of radius $a$, and versed sine $x$.

COR. 3. The fluent of $\dfrac{a^2\dot{t}}{a^2+t^2}$ is a circular arc, whose radius is $a$, and tangent $t$.

COR. 4. The fluent of $\dfrac{a^2\dot{s}}{s\times\sqrt{s^2-a^2}}$ is a circular arc, whose radius is $a$, and secant $s$.

COR. 5. The fluent of $\dfrac{-a\dot{x}}{\sqrt{a^2-x^2}}$ is a circular arc, whose radius is $a$, and cosine $x$.

(45.) Hence, since the circumferences of circles and corresponding arcs are as their radii, if we know the length of an arc to any one radius, we may find it for any other. Let $A=$ the length of an arc to radius 1; required to find the length of an arc subtending the same angle to radius $a$; the proportion is $1 : a :: A :$ the arc required $=a\times A$. Thus, if the length of an arc of 30° to radius $1=0{,}5235987$, the length of the corresponding arc to radius $a=a\times 0{,}5235987$.

## ON THE CORRECTION OF FLUENTS.

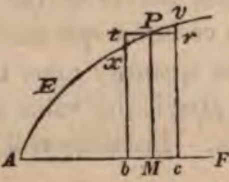
(46.) The fluxion of $x$ is $\dot{x}$; and the fluxion of $x \pm a$ is $\dot{x}$, whatever be the value of the constant part $a$, and whatever its sign. Under different circumstances, therefore, the fluent of $\dot{x}$ may be either $x$, or $x \pm a$. So that though a fluent can only have one fluxion, yet a fluxion, in different cases, may have different fluents; and this must be determined from the nature of the problem. First, take the fluent according to the rules, and observe whether this fluent becomes equal to nothing, or to some determined value, when the nature of the problem requires that it should; if it do, no constant quantity is to be annexed to it; if not, some constant quantity must be added or subtracted to make it $=0$, or to give it the value required. This is called the correction of a fluent. Instances will be given in the following sections.

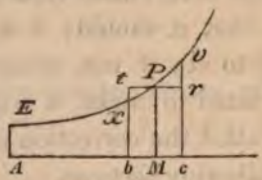
## CHAP. VI.

### ON THE AREAS OF CURVES.

(47.) THE fluxion of the area of an algebraic curve is equal to the rectangle contained by the ordinate and the fluxion of the abscissa.

Let $AMP$ be any curvilinear area, generated by the uniform motion of the ordinate $PM$. Take $Mb$ and $Mc$ on each side of $MP$, and equal to each other; and let $tPr$, parallel to $AF$, meet $bx$ and $cv$, which are drawn parallel to $MP$, in $t$ and $r$. Then, since the abscissa $AM$ is supposed to flow uniformly, if the ordinate $MP$ be conceived to increase, either of the equal parallelograms $PtbM$, $PrcM$, is greater than the preceding increment $PxbM$, and less than the succeeding $PvcM$. Also, since $PM=rc$, the rectangle $PMcr$ increases or decreases according to the increase or decrease of $MC$; therefore $PMcr$ is the fluxion of the area $AMP$. Hence, if $AM=x$, $PM=y$, the fluxion of the area $APM=y\dot{x}$.

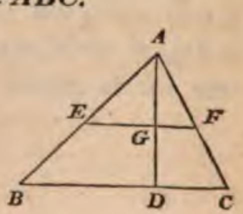




### EXAMPLES.

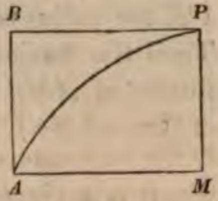
(48.) Ex. 1. To find the area of a triangle $ABC$.

Draw $AD$ perpendicular, and $EF$ parallel to $BC$. Let $AD=a$, $BC=b$, $AG=x$, $EF=y$; then, by similar triangles, $a : b :: x : y$; $\therefore y = \dfrac{bx}{a}$; $\therefore y\dot{x}$, the fluxion of the



area $= \frac{bx\dot{x}}{a}$, and the fluent $= \frac{bx^2}{2a} = \frac{y \times x}{2}$; that is, the area $ABC = \frac{BC \times DA}{2}$.

Ex. 2. To find the area of the common parabola *APM*.



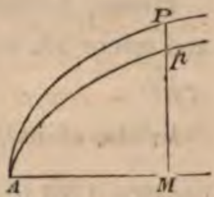
The equation is $ax = y^2$; $\therefore a\dot{x} = 2y\dot{y}$; $\therefore y\dot{x} = \frac{2y^2\dot{y}}{a}$; and the fluent $= \frac{2y^3}{3a} = \frac{2y^2 \times y}{3a} = \frac{2ax \times y}{3a} = \frac{2}{3}yx$, or the area $AMP = \frac{2}{3}$ $AM \times MP$, $= \frac{2}{3}$ of the circumscribing rectangle *AMPB*.

Ex. 3. In general, to find the area of any parabola. The general equation is $a^{n-1}x = y^n$;

$\therefore a^{n-1}\dot{x} = ny^{n-1}\dot{y}$; $\therefore y\dot{x} = \frac{ny^n\dot{y}}{a^{n-1}}$; and the fluent $= \frac{ny^{n+1}}{\overline{n+1}.a^{n-1}}$ $= \frac{ny^n \times y}{\overline{n+1}.a^{n-1}} = \frac{na^{n-1}x \times y}{\overline{n+1}.a^{n-1}} = \frac{n \times xy}{n+1} = \frac{n}{n+1} \times$ the circumscribing rectangle.

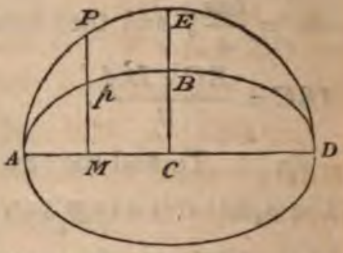
Cor. If $n = 2$, the area $= \frac{2}{3}$ of this rectangle, as before.

Ex. 4. To compare the areas of two parabolas described on the same axis, whose latera recta are *L* and *l*.



Let *AP* and *Ap* be the parabolas; *AM* the common axis. Then the fluxion of *AMP* : the fluxion of *AMp* :: $MP \times \dot{x} : Mp \times \dot{x}$ :: $MP : Mp$; but $L \times AM = MP^2$; $\therefore \sqrt{L \times AM} = MP$; and $\sqrt{l \times AM} = Mp$; $\therefore$ the fluxion of *AMP* : the fluxion of *AMp* :: $\sqrt{L \times AM} : \sqrt{l \times AM} :: \sqrt{L} : \sqrt{l}$. But quantities, whose fluxions are in constant ratio, are themselves in the same constant ratio; $\therefore$ the area *AMP* : the area *AMp* :: $\sqrt{L} : \sqrt{l}$.

Ex. 5. To compare the area of an ellipse, whose major axis is $2a$, and minor $2b$, with the area of a circle, whose diameter is the major axis of the ellipse.



Let *APD* be the circle, *ApD* the ellipse, *AM* an abscissa common to both, *PM* and *pM* the corresponding ordinates, *CB* half the minor axis of the ellipse, and *CE* the radius of the circle $= CA$. Then the fluxion of *AMP* : the fluxion of *AMp* :: *MP* : *Mp* :: *CE* : *CB* (by the nature of the ellipse) :: *CA* : *CB* :: 2*CA* : 2*CB*.
:: the major axis : the minor, which is a given ratio;

∴ the area *AMP* : the area *AMp* :: the major axis : the minor; and *ACE* : *ACB*, or the whole area of the circle : the whole area of the ellipse :: the major : the minor.

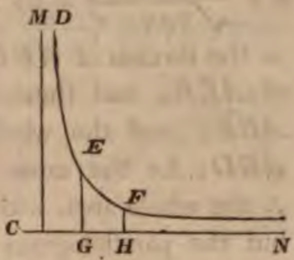
(49.) The same proposition may be easily proved by means of the following Lemma. (See the FIGURE above.)

If *APD* be a circle, *AD* the diameter $= 2a$, and *PM* an ordinate, the area *AMP* = the fluent of $\dot{x} \times \sqrt{2ax - x^2}$. For *PM* or $y = \sqrt{2ax - x^2}$; $\therefore y\dot{x}$ or the fluxion of the area *AMP* $= \dot{x} \times \sqrt{2ax - x^2}$; ∴ the area *AMP* itself = the fluent of $\dot{x} \times \sqrt{2ax - x^2}$.

Hence, to find the area of an ellipse, whose major axis is $2a$ and minor $2b$, in terms of the area of a circle whose diameter $= 2a$.

In the ellipse, $y = \frac{b}{a} \times \sqrt{2ax - x^2}$; $\therefore y\dot{x}$ or the fluxion of the area *AMp* (See FIG. to Ex. 5.) $= \frac{b}{a} \times \dot{x} \times \sqrt{2ax - x^2}$; ∴ the fluent or area $AMp = \frac{b}{a} \times$ the fluent of $\dot{x} \times \sqrt{2ax - x^2} = \frac{b}{a} \times$ the circular area *AMP*, and the whole area of the ellipse $ABD = \frac{b}{a} \times$ the area of the circle *AED*; therefore,
the area of the ellipse : area of the circle on major axis :: $b$ : $a$, or :: the minor axis : the major.

Ex. 6. Let $DEF$ be an hyperbola, of which the asymptotes are $CM$ and $CN$; to find the area $EGHF$ comprehended between two ordinates $EG$, $FH$.

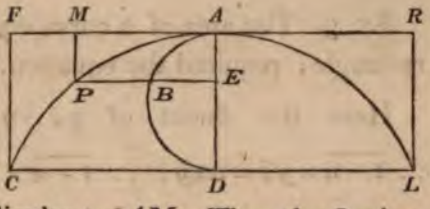


Let $CG=a$, $GE=b$, $GH=x$, $HF=y$; then, by the nature of the hyperbola, $CG\times GE=CH\times HF$, or $a\times b=\overline{a+x}\times y$; $\therefore\ y=\dfrac{ab}{a+x}$, and $y\dot{x}=\dfrac{ab\times\dot{x}}{a+x}$; $\therefore$ the fluent $=ab\times$ hyp. log. $\overline{a+x}+$ cor.

Let the area $=0$, $x=0$; $\therefore$ cor. $=-ab\times$ hyp. log. $a$; $\therefore$ the correct area $EGHF=ab\times$ hyp. log. $\dfrac{a+x}{a}$.

Cor. If $CG$ and $GE$ each $=1$, $y\dot{x}=\dfrac{\dot{x}}{1+x}$; and the fluent $=$ hyp. log. $\overline{1+x}+$ cor. $=$ hyp. log. $\overline{1+x}-$ hyp. log. 1; but hyp. log. of $1=0$; $\therefore\ EGFH=$ hyp. log. $\overline{1+x}$; or the area $EGHF$ is the hyperbolic logarithm of the abscissa; the modulus here $=1$.

Ex. 7. To find the area of a cycloid.



Let $CAL$ be a cycloid, $AD$ the axis, $ABD$ the generating circle, $AF$ a tangent at the vertex, $CF$ parallel to $AD$. Take any point $P$ in the arc, and draw $PM$ perpendicular to $AM$. Then the fluxion of the external area $AMP=PM\times$ the fluxion of $AM$. Let $AE=x$, $AD=2a$; then $BE=\sqrt{2ax-x^2}$, and the fluxion of $BE=\dfrac{a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}$.

Also $PB=$ the arc $BA$; $\therefore$ the fluxion of $PB=\dfrac{a\dot{x}}{\sqrt{2ax-x^2}}$; $\therefore$ the fluxion of $PB+BE$, or of $AM=\dfrac{2a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}$; and the fluxion of the area $APM$ or $PM \times$ the fluxion of $AM = \frac{2ax\dot{x} - x^2\dot{x}}{\sqrt{2ax - x^2}} = \dot{x} \times \frac{2ax - x^2}{\sqrt{2ax - x^2}} = \dot{x} \times \sqrt{2ax - x^2}$; but $\dot{x} \times \sqrt{2ax - x^2}$ (Art. 49.) $=$ the fluxion of $AEB$; $\therefore$ the fluxion of $APM =$ the fluxion of $AEB$; and these two areas begin together; $\therefore APM = ABE$; and the whole external area $ACF =$ the semicircle $ABD$; for the same reason, $ALR =$ the other semicircle; $\therefore$ the whole area, without the cycloid, $=$ the generating circle; but the parallelogram $CFRL = CL \times DA =$ the circumference $\times$ the diameter $= 4$ generating circles; $\therefore$ the area of the cycloid $= 3$ generating circles.

Ex. 8. To find the area of a curve, whose equation is $y = \frac{ax}{\sqrt{ax - x^2}}$.

Here $y\dot{x} = \frac{ax\dot{x}}{\sqrt{ax - x^2}}$; whose fluent is (Fluent 23.) $a \times$ a circular arc of radius $\frac{1}{2}a$, and versed sine $x - a \times \sqrt{ax - x^2}$, which vanishes when $x = 0$; and if $x = a$, it becomes $a \times$ the semi-circumference, whose radius $= \frac{1}{2}a =$ the area of two circles of radius $= \frac{1}{2}a$.

Ex. 9. The area of a curve equals $n$ times its circumscribing rectangle; required the equation.

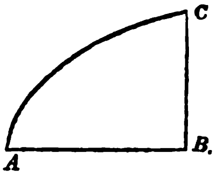
Here the fluent of $y\dot{x} = n \times yx$; $\therefore y\dot{x} = ny\dot{x} + nx\dot{y}$; $\therefore \overline{1-n} \times y\dot{x} = nx\dot{y}$; $\therefore \overline{1-n} \times \frac{\dot{x}}{x} = n \times \frac{\dot{y}}{y}$; and $\overline{1-n} \times$ hyp. log. $x = n \times$ hyp. log. $y$; or $x^{1-n} \propto y^n$.

In the common parabola, $n = \frac{2}{3}$; $\therefore x^{\frac{1}{3}} \propto y^{\frac{2}{3}}$, or $x \propto y^2$, and $ax = y^2$ the equation.

Ex. 10. Required the nature of a curve whose area $=$ hyp. log. $\frac{a+x}{a-x}$.

Here the fluent of $y\dot{x} =$ hyp. log. $\dfrac{a+x}{a-x} =$ hyp. log. $a+x-$ hyp. log. $a-x$;

$\therefore\ y\dot{x} = \dfrac{\dot{x}}{a+x} + \dfrac{\dot{x}}{a-x} = \dfrac{2a\dot{x}}{a^2-x^2}$; and $y = \dfrac{2a}{a^2-x^2}$ the equation.

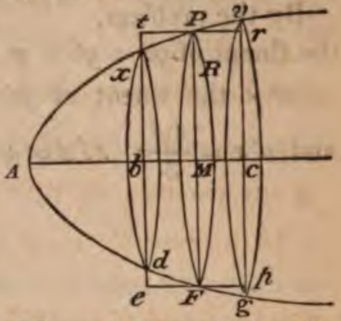


Ex. 11. Required the nature of the curve, the square of whose ordinate $BC$ is a mean proportional between some constant quantity $a^2$, and the curvilinear area $ABC$.

By the problem,
the fluent of $y\dot{x} : y^2 :: y^2 : a^2$;
$\therefore\ a^2 \times$ the fluent of $y\dot{x} = y^4$ [illegible] $a^2 y\dot{x} = 4y^3\dot{y}$; $\therefore\ a^2\dot{x} = 4y^2\dot{y}$, and $a^2 x = \dfrac{4y^3}{3}$; $\therefore\ x \propto y^3$.

# CHAP. VII.

## TO FIND THE CONTENTS OF SOLIDS.



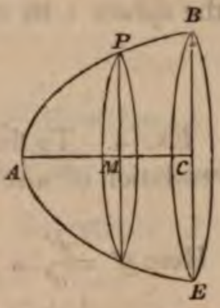
(50.) LET $PAF$ represent a solid, which is conceived to be generated by a circle, beginning its motion at $A$, and perpetually changing its magnitude, while its centre moves uniformly on the line $AM$. Let $PRF$ denote the position of this circle, corresponding to the ordinate $PM$ of the curve $AP$. Through $P$ and $F$ draw two lines, $tPr$, $eFp$, parallel to $AC$; take $Mb=Mc$, and through $b$ and $c$ draw two lines parallel to $PF$, meeting $tPr$, $eFg$, in $t$, $e$, $r$ and $p$. Conceive circles to be described on the diameters $te$, $xd$, $vg$ and $rp$. Then the cylinder $PrpF = PteF$; and $PrpF$ is less than $PvgF$; and its equal $PteF$ is greater than $PxdF$; that is, the cylinder $PrpF$ is greater than the preceding, and less than the succeeding increment of the solid. Also, since $PF$ is constant in the cylinder $PFpr$, the increase or decrease of this cylinder will vary as the increase or decrease of $MC$; but $AM$ flows uniformly, ∴ the cylinder $PFHR$ pr is the fluxion of the solid $PAF$.

Let $AM=x$; $PM=y$; $MC=\dot{x}$; then, if $p=3.14159$, &c. the area of a circle, whose radius is 1, the content of this cylinder, or the fluxion of the solid $=py^2\dot{x}$.

COR. Whatever be the form of the generating plane, as a triangle, a square, a parallelogram, &c. the fluxion of the solid will be equal to the product of this area and the fluxion of the abscissa.

(51.) **Examples.**

Ex. 1. Let $ABE$ be the solid generated by the revolution of the common parabola about its axis.



Here $y^2 = ax$; $\therefore py^2\dot{x} = pax\dot{x}$, and the fluent or content $= \frac{pax^2}{2} +$ correct.; let $x = 0$; then the content $= 0$, $\therefore C = 0$; or the content $= \frac{pax^2}{2} = \frac{pax \times x}{2} = \frac{py^2 \times x}{2}$; that is, the whole content of $ABE = p \times \frac{BC^2 \times AC.}{2}$

Cor. Since $p \times BC^2 \times AC =$ the content of the circumscribing cylinder, the paraboloid = half the circumscribing cylinder.

Ex. 2. Let $ABE$ be the solid, generated by the revolution of any parabola about its axis.

In this case, $a^{n-1}x = y^n$; $\therefore a^{n-1}\dot{x} = ny^{n-1}\dot{y}$; $\therefore py^2\dot{x} = \frac{npy^{n+1}\dot{y}}{a^{n-1}}$, and the fluent or content $= \frac{npy^{n+2}}{\overline{n+2} \times a^{n-1}} +$ correction. But if $y = 0$, the content $= 0$, $\therefore$ cor. $= 0$; $\therefore$ the content $= \frac{npy^{n+2}}{\overline{n+2}.a^{n-1}} = \frac{npy^n \times y^2}{\overline{n+2}.a^{n-1}} = \frac{npa^{n-1}xy^2}{\overline{n+2}.a^{n-1}} = \frac{n}{n+2} \times py^2x.$

Cor. The content of $ABE$ in this case $= \frac{n}{n+2} \times$ the content of the circumscribing cylinder.

If $n = 2$, the paraboloid $= \frac{1}{2}$ the cylinder as before.

Ex. 3. To find the content of a sphere.

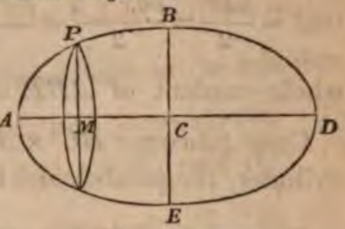
Let the radius $= a$, the versed sine $= x$, and the right sine $y$; then the equation is $y^2 = 2ax - x^2$, $\therefore py^2\dot{x} = p \times \overline{2ax\dot{x} - x^2\dot{x}}$; and the fluent $= p \times \overline{ax^2 - \frac{x^3}{3}} +$ cor. Let $x = 0$; then the sphere $= 0$; $\therefore$ cor. $= 0$; and the fluent $= p \times \overline{ax^2 - \frac{x^3}{3}}$; let $x = 2a$; then the whole content $= p \times \overline{4a^3 - \frac{8a^3}{3}} = \frac{4pa^3}{3}$.

Cor. The content of a cylinder circumscribing the sphere $=pa^2 \times 2a = 2pa^3$; therefore,

the sphere : its circumscribing cylinder :: $\frac{4pa^3}{3} : 2pa^3$,

$:: 2 : 3$.

Ex. 4. To find the content of a spheroid generated by the revolution of a semi-ellipse about the axis major.



Here $y^2 = \frac{b^2}{a^2} \times \overline{2ax - x^2}$;

$\therefore py^2\dot{x} = \frac{pb^2}{a^2} \times \overline{2ax\dot{x} - x^2\dot{x}}$;

$\therefore$ by the last case, the content of the section corresponding to the abscissa $x = \frac{pb^2}{a^2} \times \overline{ax^2 - \frac{x^3}{3}}$; the cor. $=0$; and the content of the whole spheroid $= \frac{pb^2}{a^2} \times \frac{4a^3}{3} = \frac{4pb^2a}{3}$.

Cor. 1. If the ellipse revolve round the minor instead of the major axis, since the same property of the curve obtains in each case, the content of the spheroid thus generated $= \frac{4pa^2b}{3}$. Hence the solid generated round the major : the solid generated round the minor $:: \frac{4pb^2a}{3} : \frac{4pa^2b}{3} :: b : a :: BC : CA$.

Cor. 2. Since $\frac{4pb^2a}{3} : \frac{4pa^2b}{3} :: \frac{4pa^2b}{3} : \frac{4pa^3}{3}$; $\therefore$ the solid generated round the minor is a mean proportional between the solid generated round the major and the sphere, whose diameter is equal to the major axis.

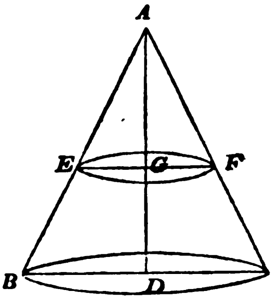
Ex. 5. To find the content of an hyperboloid.

Let $BAE$ (See the Figure in the preceding page) represent a hyperboloid, whose major axis $= 2a$, minor $= 2b$, $AM = x$, $MP = y$, $AC = e$, then $y^2 = \frac{b^2}{a^2} \times \overline{2ax + x^2}$; $\therefore py^2\dot{x} = \frac{pb^2}{a^2} \times \overline{2ax\dot{x} + x^2\dot{x}}$;

and the fluent $= \frac{pb^2}{a^2} \times \overline{ax^2+\frac{x^3}{3}}$, which needs no correction; or the whole content $= \frac{pb^2}{a^2} \times \overline{ae^2+\frac{e^3}{3}}$.

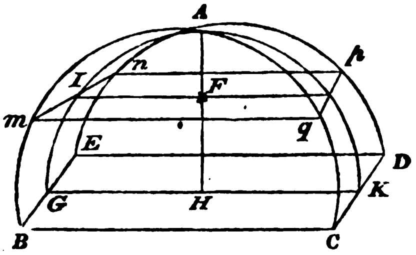
Cor. The content of a cylinder, the radius of whose base is $PM$, and altitude $AM = p \times PM^2 \times AM = \frac{pb^2}{a^2} \times \overline{2ax^2+x^3}$; ∴ the content of the hyperboloid of the altitude $x$ : the content of a cylinder of the same base and altitude :: $ax^2+\frac{x^3}{3} : 2ax^2+x^3$. Let $x$ be very small, and this becomes the ratio of $ax^2 : 2ax^2$, or of 1 : 2, which is the ratio in the paraboloid.

Ex. 6. To find the content of a cone $ABC$. Let $AD$ the axis $=a$, $DC=b$, $AG=x$, $GF=y$. Then $y=\frac{bx}{a}$; ∴ $py^2\dot{x} = \frac{pb^2x^2\dot{x}}{a^2}$, and the fluent $=\frac{pb^2x^3}{3a^2}$, which needs no correction. Let $x=a$; then the content of the whole cone $= \frac{pb^2a}{3} = \frac{1}{3}$ of a cylinder of the same base and altitude.



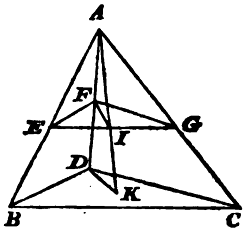
Ex. 7. To find the content of the solid called The Groin, which is generated by a variable square, $mnpq$, moving parallel to itself, the section $GAK$ through the middle of the opposite sides being semi-circular.

Let $AH=a$; $AF=x$; $FI=y$; then $y=\sqrt{2ax-x^2}$ by the nature of the circle; ∴ $2y = 2 \times \sqrt{2ax-x^2}$; ∴ the area of the generating plane $mnpq$, which answers to $py^2$ in the other cases, $= 4 \times \overline{2ax-x^2}$; hence $py^2\dot{x} = 8ax\dot{x} - 4x^2\dot{x}$; and the fluent $= 4ax^2$



$-\frac{4x^3}{3}$ + correction, and the correction $=0$. Let $x=a$; then the whole solid $BAD = 4a^3 - \frac{4a^3}{3} = \frac{8a^3}{3}$.

Ex. 8. To find the content of a pyramid, the section of which parallel to the base is any given figure.



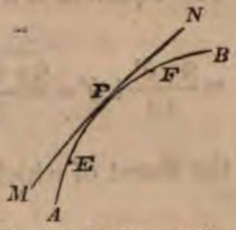
Let $EFG$ be a section parallel to the base; draw $AIK$ perpendicular to the plane $BDC$, cutting $EFG$ in $I$; join $KD$, $FI$; then the area of the base $BDC$ : the area of $EFG$ :: $DC^2 : FG^2$ :: $AD^2 : AF^2$ :: $AK^2 : AI^2$; hence if $A=$ the area of the base, $AK=a$, $AI=x$, the generating area, $EFG=\frac{Ax^2}{a^2}$; $\therefore py^2\dot{x}=\frac{Ax^2\dot{x}}{a^2}$; and the fluent $=\frac{Ax^3}{3a^2}$, which needs no correction. Let $x=a$; then the content of the whole figure $=\frac{A\times a}{3}=\frac{1}{3}$ the content of a prismatic figure of the same base and altitude.

# CHAP. VIII.

## ON THE LENGTHS OF CURVES.

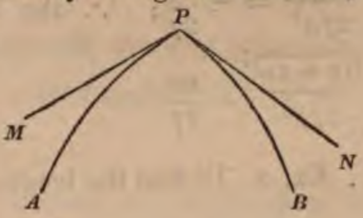
(52.) DEF. A CURVE is described by the motion of a point which is continually changing its direction.

COR. 1. If *MPN* be a tangent to the curve *AB* at *P*, the direction in which the describing point moves, when at *P*, is in the tangent *PN*.



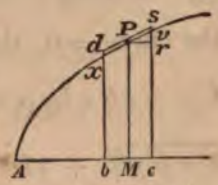
COR. 2. Its direction at any other points, *E* and *F*, is not in the line *PN*, but in tangents drawn to the curve at *E* and *F*.

(53.) If the point do not continually change its direction, but move, by a sudden change at *P*, from the curve *AP*, to describe *PB*, there may be two tangents at the point *P*, *PM* and *PN*. This is not a curve of "continued curvature," but is, as it were, broken at *P*. NEWTON's expression, "*Curvatura continua*," (Sect. 1. Lemma 6.) is applicable to those curves alone, which are described by the motion of a point continually changing its direction.



(54.) To find the length of a curve, which is referred to an axis.

Suppose *AP* the curve, *AM* the axis; *bx*, *PM*, *cv*, three equidistant ordinates. Let *dPs* be a tangent at *P*, meeting *bx* and *cv* produced in *d* and *s*. Then the direction of the point *P*, which describes the curve, when the ordinate is in the position *MP*, is in the tangent *Ps*.



Suppose the ordinate *PM*, and the abscissa *AM*, to flow uniformly for a small space, and their fluxions to be *Mc* and *rs*; then, by Art. 44, *Ps* is the fluxion of *AP*. Let $AP = z$, $AM = x$, $MP = y$, $Pr = \dot{x}$, $rs = \dot{y}$; then $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$.

## (55.) EXAMPLES.

Ex. 1. To find the length of a semicubical parabola.

The equation is $ax^2 = y^3$; $\therefore x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, and $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$; $\therefore \dot{x}^2 + \dot{y}^2$ $= \frac{9y\dot{y}^2}{4a} + \dot{y}^2 = \frac{\overline{9y+4a} \times \dot{y}^2}{4a}$; hence $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\overline{9y+4a}|^{\frac{1}{2}}}{2a^{\frac{1}{2}}} \times \dot{y}$; the fluent of which, by Art. 39, $= \frac{\overline{9y+4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$ + corr. Let the arc = 0; then $y = 0$; $\therefore 0 = \frac{\overline{4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} + C$; $\therefore C = 0 - \frac{\overline{4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} =$ $\frac{-8a^{\frac{3}{2}}}{27a^{\frac{1}{2}}} = -\frac{8a}{27}$; $\therefore$ the whole length, thus corrected, $=$ $\frac{\overline{9y+4a}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27}$.

Ex. 2. To find the length of a common parabola.

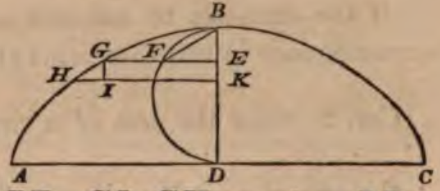
Here $ax = y^2$, and $\dot{x} = \frac{2y\dot{y}}{a}$; $\therefore \dot{x}^2 + \dot{y}^2 = \frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2 = \overline{\frac{4y^2}{a^2} + 1}$ $\times \dot{y}^2$. Let $\frac{2}{a} = \frac{1}{b}$; then $\frac{4}{a^2} = \frac{1}{b^2}$; $\therefore \dot{z}^2 = \overline{\frac{y^2}{b^2} + 1} \times \dot{y}^2 = \frac{\overline{y^2 + b^2}}{b^2} \times \dot{y}^2$; $\therefore \dot{z} = \frac{\sqrt{y^2+b^2} \times \dot{y}}{b}$, of which the fluent, (Fluent 58.) $=$ $\frac{1}{2b} \times \overline{y^4 + b^2y^2}|^{\frac{1}{2}} + \frac{1}{2}b \times \text{hyp. log.}\ \overline{y + \sqrt{y^2+b^2}}$ + corr. Let the curve $=0$, then $y = 0$; $\therefore 0 = \frac{1}{2}b \times \text{hyp. log.}\ b + C$, and $C = -\frac{1}{2}b \times \text{hyp. log.}\ b$; $\therefore$ the whole corrected length $= \frac{1}{2b} \times$ $\overline{y^4 + b^2y^2}|^{\frac{1}{2}} + \frac{1}{2}b \times \text{hyp. log.}\ \overline{y + \sqrt{y^2+b^2}} - \frac{1}{2}b \times \text{hyp. log.}\ b =$

$$\frac{1}{2b} \times \overline{y^4+b^2y^2}|^{\frac{1}{2}} + \tfrac{1}{2}b \times \text{hyp. log.}\, \overline{y+\sqrt{y^2+b^2} - \text{hyp. log. } b} =$$

$$\frac{1}{2b} \times \overline{y^4+b^2y^2}|^{\frac{1}{2}} + \tfrac{1}{2}b \times \text{hyp. log.}\, \frac{y+\sqrt{y^2+b^2}}{b}.$$

Instead of reasoning in all cases from the expression $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, it is sometimes useful to adopt other methods, according to the nature of the curve. Some instances are given in the following Examples.

Ex. 3. To find the length of the cycloid *ABC*.



Take $BD=a$, $BE=x$, $BG=z$, *GH* in the direction of a tangent at $G=\dot{z}$; draw *HIK* parallel to *GE*, and let $EK=\dot{x}$. Then, by similar triangles, *BEF*, *GIH*, $BE : BF :: GI : GH$; or, since $BF = \sqrt{DB \times BE} = \sqrt{ax}$, $x : a^{\frac{1}{2}}x^{\frac{1}{2}} :: \dot{x} : \dot{z}$; $\therefore \dot{z} = \frac{a^{\frac{1}{2}}\dot{x}}{x^{\frac{1}{2}}} = a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$, and $z = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$ + correction. Let $x=0$, $z=0$, $\therefore C=0$; and $z = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$; that is, $BG=2BF$; $\therefore BA=2BD$, and the whole arc of the cycloid $ABC=4BD$.

Ex. 4. To find the length of a circular arc, by summing the four first terms of the series, which expresses the length of the arc in terms of the tangent.

Here, according to the notation in Art. 44, $\dot{z} = \frac{a^2\dot{t}}{a^2+t^2}$; or, if $a^2\dot{t}$ be actually divided by $a^2+t^2$, $\dot{z} = \dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6} +$ &c.; $\therefore z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} +$ &c., which needs no correction. Let the arc be 45°, then $t=a$; $\therefore$ an arc of $45° = a - \frac{a}{3} + \frac{a}{5} - \frac{a}{7} +$ &c.; or, if we take four terms, the arc $= a \times \overline{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}} = \frac{76a}{105}$; and, if for $a$ we write $\frac{d}{2}$, $d$ being the diameter, the arc of $45° = \frac{38d}{105}$; ∴ the whole circumference $= \frac{304d}{105}$; or, the circumference : the diameter :: 3 : 1 nearly.

Cor. 1. If we assume $z$ an arc of 30°, and $a = 1$, then $t = \frac{1}{\sqrt{3}} = 0.5773502$; if this value be substituted for $t$, and twelve terms of the series be taken, $z$ the arc of $30° = 0.5235987$. Multiply this by 12; then the whole circumference $= 6.2831804$, where radius is 1; but the circumferences of circles are as their radii; ∴ if the diameter be assumed $= 1$, or radius $= \frac{1}{2}$, the whole circumference in this case $= 3.141590$, &c.

Cor. 2. Since the area of a circle $= \frac{\text{radius} \times \text{circumference}}{2}$, ∴ the area of a circle, whose radius is 1, $= \frac{1 \times 6.2831804}{2} = 3.14159$, &c.

Ex. 5. Compare the circumference of a circle with its diameter, by summing the four first terms of a series expressing the length of a circular arc in terms of the sine.

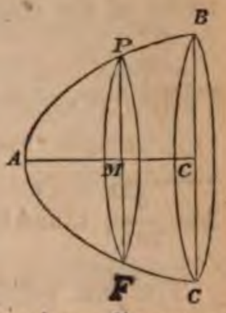
Here, according to the notation, (Art. 44.) $\dot{z} = \frac{a\dot{y}}{\sqrt{a^2 - y^2}}$, of which the fluent $= y + \frac{y^3}{2.3a^2} + \frac{3y^5}{2.4.5a^4} + \frac{15y^7}{2.4.6.7a^6} +$ &c. (Fluent 93.) Let $y = a$, the arc = a quadrant; ∴ a quadrant $= a \times : 1 + \frac{1}{6} + \frac{3}{40} + \frac{15}{336}$; or, by adding the three first terms together, and dividing the numerator and denominator of the first by 2, and of the last by 3, it $= a \times : \frac{149}{120} + \frac{5}{112} = a \times \frac{2161}{1680}$; ∴ the whole circumference $= a \times \frac{8644}{1680}$; or, if $a = \frac{d}{2}$, $d$ being the diameter, it $= \frac{d \times 4322}{1680}$; ∴ the circumference : the diameter :: 4322 : 1680 nearly. If a greater number of terms were taken, the approximation would be more correct.

(56.) Nearly in the same manner, an approximation may be made to the value of the circumference in terms of the diameter, by taking the sum of a few of the first terms of the several series which express the length of a circular arc, in terms of the cosine, versed sine, or secant. The expression for $\dot{z}$ in terms of the cosine is $\dot{z} = \frac{-a\dot{x}}{\sqrt{a^2 - x^2}} = -\dot{x} \times \overline{1 - \frac{x^2}{a^2}}\Big|^{-\frac{1}{2}}$; for the versed sine, $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - x^2}} = \frac{a\dot{x}}{\sqrt{2ax} \times \overline{1 - \frac{x^2}{2ax}}\Big|^{\frac{1}{2}}} = \frac{a^{\frac{1}{2}}\dot{x}}{\sqrt{2x}} \times \overline{1 - \frac{x^2}{2ax}}\Big|^{-\frac{1}{2}}$; and for the secant $\dot{z} = \frac{a^2\dot{s}}{s \times \sqrt{s^2 - a^2}} = \frac{a^2\dot{s}}{s^2} \times \overline{1 - \frac{a^2}{s^2}}\Big|^{-\frac{1}{2}}$.

The part, which is in the form of a residual, is to be expanded by the binomial theorem, as in the last case, and the fluent of each term to be taken separately.

# CHAP. IX.

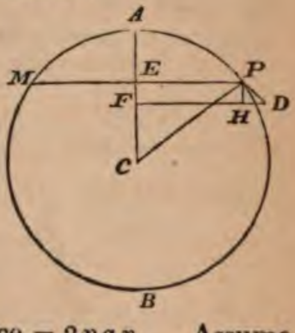
## TO FIND THE SURFACES OF SOLIDS.



(57.) LET $BAC$ represent a solid generated by the circle $PF$, as in Art. 50. The surface of the solid is generated by the circumference of that circle. Hence the fluxion of the surface will equal the circumference of $PF$ multiplied by the velocity of the point $P$, or equal the circumference $PF$ multiplied by the fluxion of $AP$. Let $PM=y$, $AP=z$, $p=3.14159$, &c. the circumference of a circle whose diameter is 1; then the fluxion of the surface $=2py\dot{z}$.

(58.) EXAMPLES.

Ex. 1. To find the surface of a sphere $ABP$.



Let $EP$, $FD$ be two ordinates, $PD$ a tangent at $P$, $PH$ perpendicular to $TD$. Then if $CP=a$, $AE=x$, $AP=z$, $EP=y$, $PH=\dot{x}$, $PD=\dot{z}$, by similar triangles $PHD$, $CPE$, $\dot{z} : \dot{x} :: a : y$; $\therefore 2py\dot{z}$, the fluxion of the surface $= 2pa\dot{x}$; and the fluent $=2pax+$ correction. Let $x=0$, the surface $=0$, and the correction $=0$; $\therefore$ the surface $=2pax$. Assume $x=2a$; then the whole surface $=4pa^2=$ the area of four great circles.

FD

[illegible]

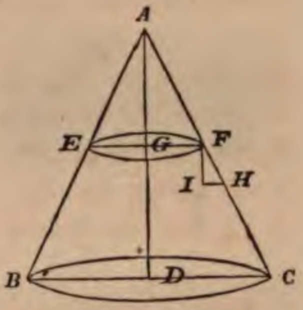
COR. The surface $MAP \propto x$.

Ex. 2. To find the surface of the solid generated by the revolution of the common parabola about its axis.

Here $ax=y^2$; $\therefore \dot{x}=\frac{2y\dot{y}}{a}$, and $\dot{x}^2=\frac{4y^2\dot{y}^2}{a^2}$; $\therefore \dot{z}^2=\frac{4y^2\dot{y}^2}{a^2}+\dot{y}^2$ $=\frac{\overline{4y^2+a^2}\times\dot{y}^2}{a^2}$; $\therefore \dot{z}=\dot{y}\times\frac{\sqrt{4y^2+a^2}}{a}$; hence, $2py\dot{z}=$

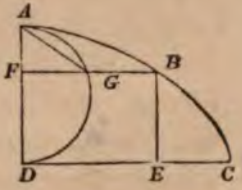
$\frac{2py\dot{y}\sqrt{4y^2+a^2}}{a}$, and the fluent $= \frac{2py\dot{y} \times \overline{4y^2+a^2}|^{\frac{3}{2}}}{8y\dot{y} \times \frac{3}{2} \times a} =$ $\frac{p \times \overline{4y^2+a^2}|^{\frac{3}{2}}}{6a}$ + correction. Let the surface $=0$, $y=0$; $\therefore$ the correction $=-\frac{pa^3}{6}$; $\therefore$ the corrected surface $= \frac{p \times \overline{4y^2+a^2}|^{\frac{3}{2}}}{6a} - \frac{pa^3}{6}$.

Ex. 3. To find the surface of a cone.



Let $AC=s$, $AF=z$, $FG=y$, $CD=b$, $FH=\dot{z}$, $HI=\dot{y}$; then, by similar triangles, $FHI$, $ACD$, $\dot{z} : \dot{y} :: s : b$; $\therefore$ $\dot{z}=\frac{s\dot{y}}{b}$; and the fluxion of the surface, or $2py\dot{z}=\frac{2ps}{b} \times y\dot{y}$. The fluent $=\frac{2ps}{b} \times \frac{y^2}{2}$, and the corr. $=0$. Let $y=b$; then the whole surface $= \frac{2psb^2}{2b} = psb = 2pb \times \frac{s}{2}$ = the circumference of the base × ½ the side $AC$.

Ex. 4. To find the surface generated by the revolution of the cycloid $ABC$ about its base $DC$.



Let $AD=a$, $AF=x$, $AB=z$, $BE=y$, $=a-x$; then $AG=a^{\frac{1}{2}}x^{\frac{1}{2}}$; $AB=2a^{\frac{1}{2}}x^{\frac{1}{2}}$; $\therefore \dot{z}=a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; $\therefore$ $2py\dot{z} = 2p \times \overline{a-x} \times a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}=2pa^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x}-2pa^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$; and the fluent $=4pa^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{4pa^{\frac{1}{2}}x^{\frac{3}{2}}}{3}$ + correction, and the correction $=0$; $\therefore$ the surface generated by $AB=4pa^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{4pa^{\frac{1}{2}}x^{\frac{3}{2}}}{3}$. Let $x=a$; then the whole surface $=4pa^2-\frac{4pa^2}{3}=\frac{8pa^2}{3}$.

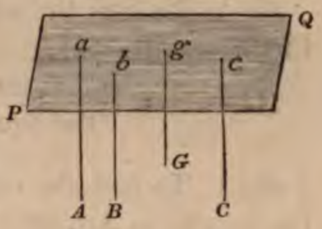
Ex. 5. To find the surface of the figure called The Groin; (Vide Art. 51. Ex. 7.)

Let $AH=a$, $AF=x$, $IF=y$, $AI=z$; then, since $\dot{z} : \dot{x} :: a : (y)\sqrt{2ax-x^2}$, $\dot{z}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}$; also, $2FI=2\times\sqrt{2ax-x^2}$; $\therefore$ $2py$ in the other cases $=8\times\sqrt{2ax-x^2}$ in this case; hence $2py\dot{z}=8\times\sqrt{2ax-x^2}\times\frac{a\dot{x}}{\sqrt{2ax-x^2}}=8a\dot{x}$; and the fluent $= 8ax +$ the correction, but the correction $=0$; $\therefore$ the surface corresponding to the abscissa $AF=8ax$. Let $x=a$, then the whole surface $=8a^2$.

# CHAP. X.

## ON THE CENTER OF GRAVITY.

### LEMMA.



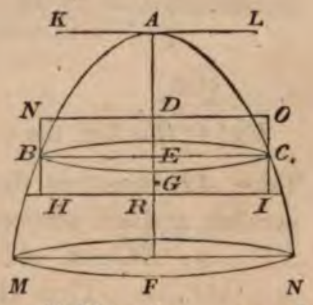
By Mechanics, if $A$, $B$, $C$, &c. be any number of bodies, whose center of gravity is $G$; and $Aa$, $Bb$, $Cc$, $Gg$, be drawn perpendicular to a plane $PQ$; then the line

$$Gg = \frac{A\times Aa + B\times Bb + C\times Cc + \text{\&c.}}{A+B+C+\text{\&c.}}.$$

The truth of this Proposition is assumed in the following articles.

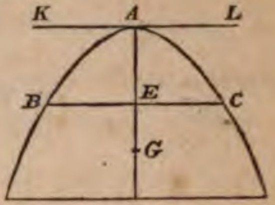
### PROPOSITION.

(59.) To find the center of gravity of a body taken as an area, solid, curve line, or surface.



Let $MAN$ be the body, $AF$ its axis; draw $KAL$, $BEC$ perpendicular to $AF$. Then, since $AF$ is the axis, the figure will balance round it, or the center of gravity is in $AF$. Conceive the body to be composed of an indefinite number of particles, and multiply each particle by its distance from $KL$. Then, by the lemma, if $G$ be the center of gravity, $AG$ is equal to the sum of all these products, divided by all the particles, or by the body itself. The sum of all these products is found by taking the fluxion of the sum in the first instance, and then the fluent. Hence, if $\dot{s}$ represent the fluxion of the body at the distance $x$ from $KL$, $AG = \dfrac{\text{the fluent of } x\dot{s}}{\text{the fluent of } \dot{s}}$.

(60.) To find the center of gravity of an area.



In this case, the fluxion of the numerator $= 2yx\dot{x}$; $\therefore AG = \dfrac{\text{the fluent of } 2yx\dot{x}}{\text{the fluent of } 2y\dot{x}}$ $= \dfrac{\text{the fluent of } yx\dot{x}}{\text{the fluent of } y\dot{x}}$.

(61.) To find the center of gravity of a solid. (See the FIGURE in Art. 59.)

Here $AG = \dfrac{\text{the fluent of } py^2\dot{x} \times x}{\text{the fluent of } py^2\dot{x}} = \dfrac{\text{the fluent of } y^2x\dot{x}}{\text{the fluent of } y^2\dot{x}}$.

(62.) To find the center of gravity of a curve line. (See the FIGURE in Art. 60.)

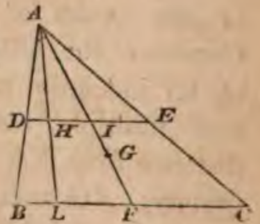
In this case, the fluxion of the numerator $= x\dot{z}$; and the fluxion of the denominator $= \dot{z}$; $\therefore AG = \dfrac{\text{fluent } x\dot{z}}{\text{fluent } \dot{z}}$.

(63.) To find the center of gravity of a surface.

The fluxion of the numerator $= 2pyx\dot{z}$, and that of the denominator $= 2py\dot{z}$; $\therefore AG = \dfrac{\text{fluent } 2pyx\dot{z}}{\text{fluent } 2py\dot{z}} = \dfrac{\text{fluent } yx\dot{z}}{\text{fluent } y\dot{z}}$.

## (64.) EXAMPLES.

Ex. 1. To find the center of gravity of a triangle $ABC$.



Take $AF$ bisecting the base; draw $DE$ parallel, and $AL$ perpendicular, to $BC$; let $AF = a$; $BC = b$, $AI = x$, $DE = y$, $AH = v$, $AL = c$. Then the center of gravity is in $AF$, and $AG = \dfrac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \dfrac{\text{fluent } DE \times AH \times \text{the fluxion of } AH}{\text{triangle } ABC}$.

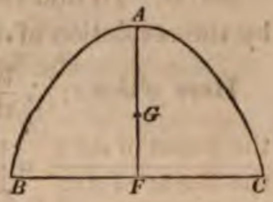
Now $AF : AL :: AI : AH$,

or, $a : c :: x : v = \frac{cx}{a}$; $\therefore \dot{v} = \frac{c\dot{x}}{a}$, and $y = \frac{bx}{a}$; $\therefore AG = \frac{\text{fluent } \frac{c\dot{x}}{a} \times \frac{bx}{a} \times x}{\text{triangle } ABC} = \frac{cb \times x^3}{3\,a^2 \times \text{triangle}} =$, for the whole triangle, $\frac{cb \times a^3}{3\,a^2 \times \frac{cb}{2}} = \frac{2a}{3}$.

Ex. 2. To find the center of gravity of the common parabola, *BAC*.

Here $y^2 = ax$; $\therefore y = a^{\frac{1}{2}}x^{\frac{1}{2}}$; $\therefore \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}}$

$= \frac{\text{fluent } a^{\frac{1}{2}}x^{\frac{3}{2}}\dot{x}}{\text{fluent } a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}} = \frac{\frac{3}{2}a^{\frac{1}{2}}x^{\frac{5}{2}}}{\frac{5}{2}a^{\frac{1}{2}}x^{\frac{3}{2}}} = \frac{3x}{5} = AG.$



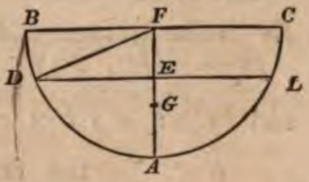
Ex. 3. To find the center of gravity of any parabola.

Here $y^n = a^{n-1}x$; $\therefore y = a^{\frac{n-1}{n}}x^{\frac{1}{n}}$; and $\frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} =$

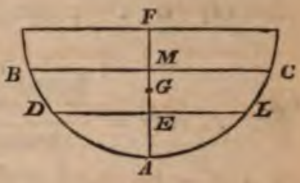
$\frac{\text{fluent } a^{\frac{n-1}{n}}x^{\frac{1}{n}+1}\dot{x}}{\text{fluent } a^{\frac{n-1}{n}}x^{\frac{1}{n}}\dot{x}} = \frac{\frac{n+1}{n}x^{\frac{2n+1}{n}}}{\frac{2n+1}{n} \times x^{\frac{n+1}{n}}} = \frac{\overline{n+1} \times x}{2n+1} = AG.$

Ex. 4. To find the center of gravity of a semi-circle.

Let $FE$ in this case $= x$, $DE = y$, $FD = r$; then $x^2 + y^2 = r^2$; $\therefore x\dot{x} = -y\dot{y}$; $\therefore \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \frac{\text{fluent } -y^2\dot{y}}{\text{fluent } y\dot{x}}$ $= \frac{\text{cor.} - \frac{1}{3}y^3}{\text{area } DEFB}$. Now when the distance of the center of gravity from $BC = 0$, $DE$ coincides with $BF$; $\therefore FG = \frac{r^3 - y^3}{3 \times \text{area } BDEF}$; or, for the whole semicircle $= \frac{r^3}{3 \times AFB}$. Corol. $AG = r - \frac{r^3}{3 \times AFB}$.



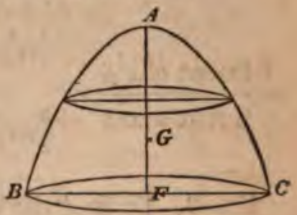
Ex. 5. To find the center of gravity of the segment of a circle *BAC.*



Take $F$ the center; join $BC$, and draw $FA$ perpendicular to $BC$; the center of gravity is in $FA$. Let $FE=x$, $ED=y$, $MB=b$; then, by the last case, $FG=\dfrac{\text{the corr.}-\frac{1}{3}y^3}{\text{the area } BMED}=\dfrac{b^3-y^3}{3\times\text{ the area } BMED}$;

$\therefore FG$ for the whole segment $=\dfrac{b^3}{3\times\text{ the area } BMA}$.

Ex. 6. To find the center of gravity of the solid generated by the revolution of the common parabola about its axis.



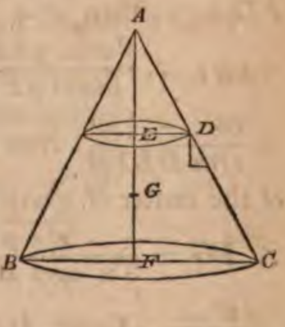
Here $y^2=ax$; $\therefore \dfrac{\text{the fluent of } y^2x\dot{x}}{\text{the fluent of } y^2\dot{x}}=\dfrac{\text{the fluent of } ax^2\dot{x}}{\text{the fluent of } ax\dot{x}}=\dfrac{\text{the fluent of } x^2\dot{x}}{\text{the fluent of } x\dot{x}}=\dfrac{2x^3}{3x^2}=\dfrac{2x}{3}=AG$.

Ex. 7. To find the center of gravity of any paraboloid.

Let $y^n=a^{n-1}x$; $\therefore y=a^{\frac{n-1}{n}}x^{\frac{1}{n}}$; $\therefore y^2=a^{\frac{2n-2}{n}}x^{\frac{2}{n}}$; and $\dfrac{\text{fluent of } y^2x\dot{x}}{\text{fluent of } y^2\dot{x}}$

$$=\frac{\text{fluent of } a^{\frac{2n-2}{n}}x^{\frac{2}{n}+1}\dot{x}}{\text{fluent of } a^{\frac{2n-2}{n}}x^{\frac{2}{n}}\dot{x}}=\frac{\overline{\frac{2}{n}+1}\,x^{\frac{2}{n}+2}}{\overline{\frac{2}{n}+2}\times x^{\frac{2}{n}+1}}=\frac{\overline{n+2}.x}{2n+2}.$$

Ex. 8. To find the center of gravity of a cone.



By similar triangles, $AED$, $AFC$, $AF : FC :: AE : ED$. Or, if $AF=a$, and $FC=b$, $a : b :: x : y=\dfrac{bx}{a}$, and $y^2=\dfrac{b^2x^2}{a^2}$; $\therefore \dfrac{\text{the fluent of } y^2x\dot{x}}{\text{the fluent of } y^2\dot{x}}=$

$$\frac{\text{the fluent of } \frac{b^2}{a^2}x^3\dot{x}}{\text{the fluent of } \frac{b^2}{a^2}x^2\dot{x}}=\frac{3x^4}{4x^3}=\frac{3x}{4}=AG.$$

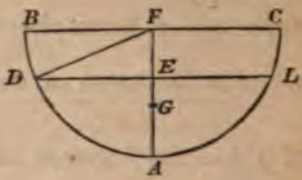
Ex. 9. To find the center of gravity of a hemisphere. Let $BAC$ (See the second FIGURE in the preceding page) represent a hemisphere; here $y^2 = 2ax - x^2$; $\therefore$ the fluent of $y^2 x\dot{x} =$ the fluent of $2ax^2\dot{x} - x^3\dot{x} = \frac{2ax^3}{3} - \frac{x^4}{4}$; and the fluent of $y^2\dot{x} =$ the fluent of $2ax\dot{x} - x^2\dot{x} = ax^2 - \frac{x^3}{3}$; $\therefore \frac{\text{the fluent of } y^2 x\dot{x}}{\text{the fluent of } y^2\dot{x}}$ $= \frac{\frac{2}{3}ax^3 - \frac{1}{4}x^4}{ax^2 - \frac{1}{3}x^3}$. Let $x = AF = a$; then $AG = \frac{\frac{2}{3}a^4 - \frac{1}{4}a^4}{a^3 - \frac{1}{3}a^3} = \frac{\frac{5}{12}a^4}{\frac{2}{3}a^3}$ $= \frac{5a}{8}$.

Ex. 10. To find the center of gravity of a hemispheroid. Let $BAC$ (See the second FIGURE in p. 57.) represent a hemispheroid; then $y^2 = \frac{b^2}{a^2} \times \overline{2ax - x^2}$; $\therefore$ the fluent of $y^2 x\dot{x} =$ $\frac{b^2}{a^2} \times$ the fluent of $\overline{2ax^2\dot{x} - x^3\dot{x}} = \frac{b^2}{a^2} \times \frac{2ax^3}{3} - \frac{x^4}{4}$; and the fluent of $y^2\dot{x} = \frac{b^2}{a^2} \times$ the fluent of $\overline{2ax\dot{x} - x^2\dot{x}} = \frac{b^2}{a^2} \times \overline{ax^2 - \frac{x^3}{3}}$; $\therefore \frac{\text{the fluent of } y^2 x\dot{x}}{\text{the fluent of } y^2\dot{x}} = \frac{\frac{b^2}{a^2} \times \frac{2}{3}ax^3 - \frac{1}{4}x^4}{\frac{b^2}{a^2} \times ax^2 - \frac{x^3}{3}}$; whence by the last case, if $x = a$, $AG = \frac{5a}{8}$.

COR. 1. Since $b$ is not found in the expression $\frac{5a}{8}$, the distance of the center of gravity from $A$ in all hemispheroids round the same major axis is the same.

COR. 2. In the same manner, if $BAC$ be an hyperboloid, and $y^2 = \frac{b^2}{a^2} \times \overline{2ax + x^2}$, $AG = \frac{8ax + 3x^2}{12a + 4x}$; that is, if $AF = C$, $AG = \frac{8aC + 3C^2}{12a + 4C}$.

Ex. 11. To find the center of gravity of the arc of a semi-circle.



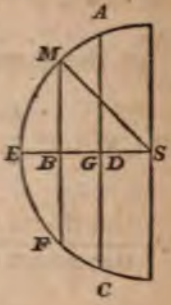
Let $FE=x$, $BD=z$, $FD=r$; then $\dot{z} : -\dot{y} :: r : x$; (Art. 44.)

$\therefore -r\dot{y}=x\dot{z}$;

$\therefore FG=\frac{\text{fluent of } x\dot{z}}{z}=\frac{\text{fluent of } -r\dot{y}}{z}$ $=\frac{-ry}{BD}$ + the corr.; and if $FG=0$, $y=r$; $\therefore FG=\frac{r^2-ry}{BD}=\frac{r^2}{BDA}$ for the arc of the quadrant; and the same is true for the other side; $\therefore FG=\frac{r^2}{BDA}$ for the arc of the semi-circle.

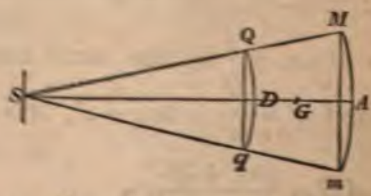
Cor. $AG=r-\frac{r^2}{BDA}$.

Ex. 12. To find the center of gravity of any arc $AEC$ of a circle.



Take $S$ the center; and draw $SDE$ perpendicular to the chord $AC$; the center of gravity $G$ is in that line. Let $AD=b$, $SE=r$, $SB=x$, $BM=y$, $AM=z$. Then $x\dot{z}=-r\dot{y}$; $\therefore$ the fluent of $x\dot{z}=-ry+$ the corr. $=rb-ry$; and $SG=\frac{\text{the fluent of } x\dot{z}}{z}=\frac{rb-ry}{AM}$; or taking the whole arc $AE$, and therefore $y=0$, $SG=\frac{r\times b}{AE}=\frac{SE\times AD}{AE}$; and the same conclusion holds for $EC$; $\therefore SG$ for the whole arc $AEC=\frac{SE\times AD}{AE}=\frac{SE\times AC}{AEC}$.

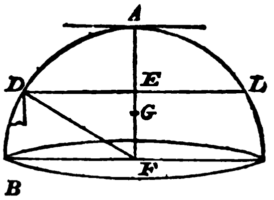
Ex. 13. To find the center of gravity of the sector of a circle.



Let $S$ be the center of the circle; $Mm$ the chord of the arc $MAm$; $SA$ perpendicular to $Mm$; $QDq$ an arc at any distance $SD$; $Qq$ its chord. Take $MAm=c$, $SM=r$, $Mm=a$, $SQ=x$, $QDq=v$; then $r : x :: a : Qq=$

$\frac{ax}{r}$; ∴ $yx\dot{x}$, in the preceding cases, corresponds in this instance with $\frac{ax}{r} \times x\dot{x} = \frac{ax^2\dot{x}}{r}$; ∴ $SG = \frac{\text{the fluent of } yx\dot{x}}{\text{the fluent of } y\dot{x}} = \frac{2}{r} \times \frac{\text{the fluent of } ax^2\dot{x}}{vx} = \frac{2ax^2}{3rv}$; and for the whole sector $= \frac{2ar}{3c}$.

Ex. 14. To find the center of gravity of the surface of a hemisphere.



Let $AE = x$, $ED = y$, $AD = z$, $DF = r$; then $AG = \frac{\text{the fluent of } yx\dot{z}}{\text{the fluent of } y\dot{z}}$. Now $\dot{z} : \dot{x} :: r : y$; ∴ $y\dot{z} = r\dot{x}$; ∴ $AG = \frac{\text{the fluent of } rx\dot{x}}{\text{the fluent of } r\dot{x}} = \frac{x^2}{2x} = \frac{x}{2}$; or for the whole hemisphere $= \frac{r}{2}$.

Ex. 15. To find the center of gravity of the surface of a cone.

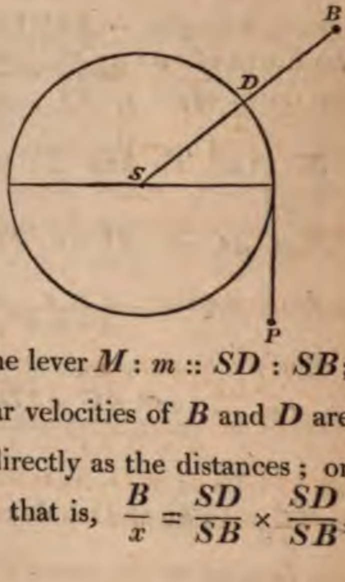
Let $AC = s$, $AF = a$, $FC = b$; (See the third FIGURE in p. 60.) then $\dot{z} : \dot{x} :: s : a$; ∴ $\dot{z} = \frac{s\dot{x}}{a}$; also $y : x :: b : a$; ∴ $y = \frac{bx}{a}$, and $AG = \frac{\text{the fluent of } yx\dot{z}}{\text{the fluent of } y\dot{z}} = \frac{\frac{s}{a} \times \frac{b}{a} \times \text{fluent of } x^2\dot{x}}{\frac{s}{a} \times \frac{b}{a} \times \text{fluent of } x\dot{x}} = \frac{2x}{3}$; ∴ for the whole conical surface $AG = \frac{2AF}{3}$.

## Chap. XI.

### ON THE CENTERS OF GYRATION, OSCILLATION, AND PERCUSSION.

### Proposition.

(65.) If $P$ represent any moving force acting at a distance $SD$ from the axis $S$, and a body $B$ revolve round the axis by the action of $P$, the same angular velocity will be produced in a given time, if a weight $= \frac{B \times SB^2}{SD^2}$ be placed in $D$.



Let $M$ and $m$ represent two moving forces acting at $B$ and at $D$; $V$ and $v$ the velocities; and $B$ and $x$ the quantities of matter at these points. Then since $M \propto Q \times V$, we have $\frac{M}{m} = \frac{B}{x} \times \frac{V}{v}$; hence $\frac{B}{x} = \frac{M}{m} \times \frac{v}{V}$. Now since the effects are the same, we have by the lever $M : m :: SD : SB$; $\therefore \frac{M}{m} = \frac{SD}{SB}$; also, since the angular velocities of $B$ and $D$ are the same, the linear velocities are directly as the distances; or $V : v :: SB : SD$; $\therefore \frac{v}{V} = \frac{SD}{SB}$; that is, $\frac{B}{x} = \frac{SD}{SB} \times \frac{SD}{SB}$, or $x = \frac{B \times SB^2}{SD^2}$.

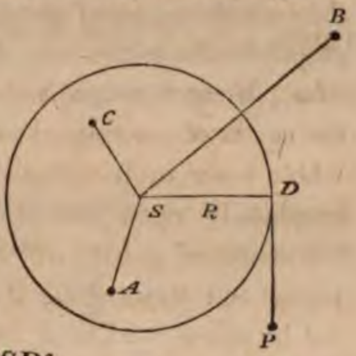
Cor. Since the accelerating force $\propto$ as the moving force divided by the quantity of matter, the accelerating force upon $D$ $\propto$ as $\frac{P}{\frac{B \times SB^2}{SD^2}} \propto \frac{P \times SD^2}{B \times SB^2}$; and the same is true for any number of bodies. The inertia of $P$ is here not taken into account.

## ON THE CENTER OF GYRATION.

(66.) Def. The center of gyration of a body, or system of bodies, is that point into which, if the whole mass were collected, a given force as $P$ applied at a given distance from the axis of suspension, would produce the same angular velocity in the same time, as if the bodies were disposed at their respective distances.

### (67.) Proposition.

To find the center of gyration of any body, or system of bodies, revolving round an axis of motion, which passes through $S$.



Let $A$, $B$, $C$, &c. be the bodies, or the particles of which the body is composed; $P$ the given force applied at $D$; $R$ the center of gyration. Then the force which accelerates $D$, whilst these bodies are at their respective distances, = (by Cor. Art. 65.) $\frac{P\times SD^2}{A\times SA^2+B\times SB^2+C\times SC^2+\&c.}$. Next, let the whole mass be collected in $R$; the accelerating force upon $D=\frac{P\times SD^2}{\overline{A+B+C+\ \&c.}\times SR^2}$. But since $P$, and the angular velocity of $D$, are by supposition the same in both cases, the absolute velocity of $D$ is the same, and $\therefore$ the accelerating force upon $D$ must be the same; that is, $\frac{P\times SD^2}{A\times SA^2+B\times SB^2+\&c.}=$

$$\frac{P\times SD^2}{\overline{A+B+C+\&c.}\times SR^2};\ \therefore SR=\sqrt{\frac{A\times SA^2+B\times SB^2+C\times SC^2+\&c.}{A+B+C+\ \&c.}}$$

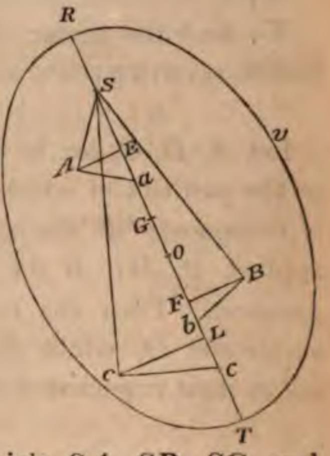
Cor. If $\dot{s}$ be the fluxion of the body at the distance $x$ from the axis, $SR=\sqrt{\frac{\text{the fluent of } x^2\dot{s}}{s}}$.

## ON THE CENTER OF PERCUSSION.

(68.) Def. The center of percussion is that point in the axis of a vibrating body, at which, if stopt by an immoveable obstacle, it would rest in equilibrio, without inclining to either side.

### (69.) Proposition.

To find the center of percussion of any body or system of bodies.



Let *RTV* represent a section of the body formed by a plane passing through the center of gravity *G*, and perpendicular to the axis of suspension passing through *S*. Let *O* be the center of percussion; suppose the whole body to be projected orthographically upon the plane *RVT*, the center of gravity will remain the same, and the angular motion will not be affected. Let *A*, *B*, *C*, &c. represent particles of the body *RVT*; join *SA*, *SB*, *SC*, and draw *Aa*, *Bb*, *Cc*, perpendicular to these lines, meeting the axis of the body in *a*, *b*, and *c*; and let fall *AE*, *BF*, *CL*, perpendicular to *ST*. The instant *O* is stopped, the particle *A* will endeavour to proceed in the direction *Aa*, with a force proportional to $A \times SA$; and this force : its force in the direction *AE*, to turn the body round $O$ :: $Aa : AE$; hence, the effect in the direction $AE = \frac{A \times SA \times AE}{Aa} = \frac{A \times SA \times SE}{SA}$ (by similar triangles) $= A \times SE$; and its effect to produce angular motion round $O = A \times SE \times aO = A \times SE \times \overline{SO - SA} = A \times SE \times SO - A \times SE \times SA = A \times SE \times SO - A \times SA^2$.

In the same manner, the effect of *B* and *C*, from which the perpendiculars cut the axis in *F* and *L*, on the other side of $O = B \times SB^2 - B \times SF \times SO$; and $C \times SC^2 - C \times SL \times SO$;

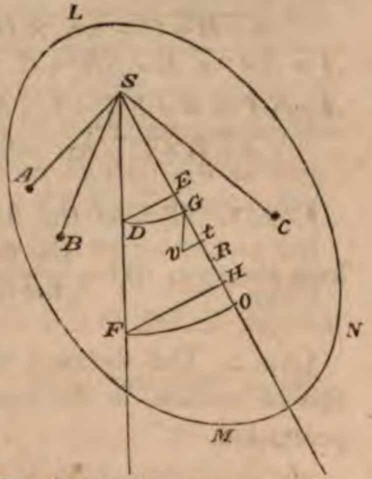
and since these forces balance each other round $O$, we have $A \times SE \times SO - A \times SA^2 = B \times SB^2 - B \times SF \times SO + C \times SC^2 - C \times SL \times SO$; $\therefore \overline{A \times SE + B \times SF + C \times SL +}$ &c. $\times SO = A \times SA^2 + B \times SB^2 + C \times SC^2 +$ &c.; or, by the nature of the center of gravity, $\overline{A+B+C+}$ &c. $\times SG \times SO = A \times SA^2 + B \times SB^2 + C \times SC^2 +$ &c.; $\therefore SO = \dfrac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \text{\&c.}}{\overline{A+B+C+}\ \text{\&c.} \times SG}$

## ON THE CENTER OF OSCILLATION.

(70.) Def. The center of oscillation is that point in a body, or system of bodies, into which, if the whole mass were collected, it would vibrate through a given angle by the force of gravity in the same time as before.

### (71.) Proposition.

To find the center of oscillation.



Let $LMN$ represent a body projected upon a plane, which is perpendicular to the axis of suspension passing through $S$. Let $SGO$ be the axis of the body at some period of its vibration, $G$ the center of gravity, and $O$ the center of oscillation. Draw $SDF$ perpendicular to the horizon; with $S$ as a center, and $SG$, $SO$, as radii, describe the circular arcs $GD$, $OF$; draw $DE$, $FH$, perpendicular to $SO$. Let $Gv$ be a line parallel to $SF$, and draw $vt$ perpendicular to $SO$; join the point $S$, and the particles $A$, $B$, $C$, of the body. Now the moving force upon $G$, whilst the particles are in the position $A$, $B$, $C$, &c. : the whole weight of the body :: $tv : Gv :: DE : SD$; $\therefore$ the moving force upon $G =$

$\overline{A+B+C+}\,\&c. \times \frac{DE}{SD \text{ or } SG}$; and this force will, in a given time, produce the same velocity, if $A$, $B$, $C$, &c. be removed, and the masses $\frac{A \times SA^2}{SG^2} + \frac{B \times SB^2}{SG^2} + \&c.$ be placed in $G$. Hence, the accelerating force upon $G$ in this case $=$ $\frac{\overline{A+B+C+\&c.} \times DE \times SG}{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}$.

Next, suppose the particles collected in $O$; the accelerating force on $O = \frac{\overline{A+B+C+\&c.} \times \frac{HF}{SF}}{A+B+C+\&c.} = \frac{HF}{SF} = \frac{DE}{SG}$, by similar triangles; and the force which accelerates $O$ : that which accelerates $G$ :: $SO$ : $SG$; $\therefore$ the force which in this case accelerates $G = \frac{DE}{SO}$. But in both instances the force at $G$ must be the same; for then, in both cases, the point $G$ will describe a small arc, and equal successive small arcs, and consequently the whole arc $GD$ in the same time. Hence, $\frac{\overline{A+B+C+}\,\&c. \times DE \times SG}{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.} = \frac{DE}{SO}$; $\therefore$ $SO =$ $\frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{\overline{A+B+C+}\,\&c. \times SG}$.

COR. 1. If $\dot{s}$ be the fluxion of the body at the distance $x$ from the axis, $SO = \frac{\text{the fluent of } x^2\dot{s}}{\text{the body} \times SG}$.

COR. 2. The distance of the center of oscillation from the axis of motion, is the same as the distance of the center of percussion.

COR. 3. $SO \times SG = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{A+B+C+\&c.}$

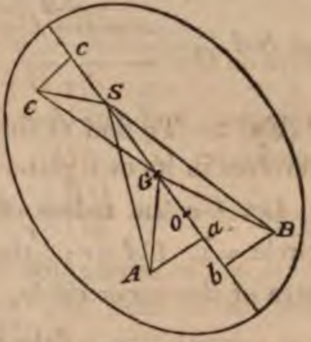
COR. 4. If $R$ represent the center of gyration,

$SG : SR :: SR : SO$.

For (Art. 67.) $SR^2 = \dfrac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{A+B+C+\&c.}$ and $SO \times SG =$ the same quantity (by the last Cor.;) $\therefore SO \times SG = SR^2$; and $SG : SR :: SR : SO$.

(72.) **Proposition.**

If the distance of the center of gravity and the axis of motion be increased, the distance of the center of gravity, and the center of oscillation, will be diminished in the same proportion.



Let $A$, $B$, $C$, &c. be particles of the body, $S$ the point of suspension, $G$ the center of gravity, and $O$ of oscillation; join $SA$, $SB$, $SC$, and $GA$, $GB$, $GC$; draw $Aa$, $Bb$, $Cc$, perpendicular to $SO$. Then,

$SA^2 = SG^2 + GA^2 + 2SG \times Ga$

$SB^2 = SG^2 + GB^2 + 2SG \times Gb$

$SC^2 = SG^2 + GC^2 - 2SG \times Gc$;

$\therefore A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c. = \overline{A+B+C+\&c.} \times SG^2 + A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c. + A \times 2SG \times Ga + B \times 2SG \times Gb - C \times 2SG \times Gc$. But, by the nature of the center of gravity, $A \times 2SG \times Ga + B \times 2SG \times Gb - C \times 2SG \times Gc = 0$; divide the whole by $\overline{A+B+C+\&c.} \times SG$; then,

$$\frac{A \times SA^2 + B \times SB^2 + C \times SC^2 + \&c.}{\overline{A+B+C+\&c.} \times SG} = \frac{\overline{A+B+C+\&c.} \times SG^2}{\overline{A+B+C+\&c.} \times SG} + \frac{A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c.}{\overline{A+B+C+\&c.} \times SG};$$

or, $SO = SG + \dfrac{A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c.}{\overline{A+B+C+\&c.} \times SG}$; here, $SO - SG$, or $GO = \dfrac{A \times GA^2 + B \times GB^2 + C \times GC^2 + \&c.}{\overline{A+B+C+\&c.} \times SG}$; that is, $GO \propto \dfrac{1}{SG}$, for the numerator, and the other part of the denominator in the same body, or system of bodies, is a constant quantity.

**Cor. 1.** $SG \times GO$ is a constant quantity.

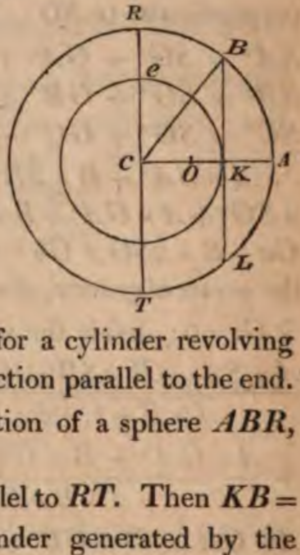
Cor. 2. If $O$ be made the point of suspension, the point $S$ will be the center of oscillation; or the center of oscillation, and the point of suspension, are convertible.

## (73.) Examples.

Ex. 1. Let the straight line $SA$ revolve about $S$; to find $O$ the center of gyration.

$$SO = \sqrt{\frac{\text{the fluent of } x^2\dot{x}}{SA}} = \sqrt{\frac{SA^3}{3SA}} = \frac{SA}{\sqrt{3}}, \text{ when } x = SA.$$

Ex. 2. To find $O$ the center of gyration of a circle, which revolves in its own plane about a center $C$.



Let $Ce$ the radius of a concentric circle $=x$, $CA=r$; then $px^2 =$ the area of the inner circle, and $2px\dot{x}$ its fluxion; $\therefore CO = \sqrt{\frac{\text{the fluent of } 2px^3\dot{x}}{px^2}} = \sqrt{\frac{px^4}{2px^2}} = \frac{r}{\sqrt{2}}$ when $x=r$.

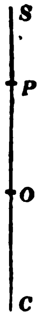
Cor. The same conclusion is true for a cylinder revolving about its axis, since it is true for every section parallel to the end.

Ex. 3. To find $O$ the center of gyration of a sphere $ABR$, revolving about its diameter $RT$.

Draw $CA$ perpendicular, and $KB$ parallel to $RT$. Then $KB = \sqrt{r^2 - x^2}$; and the surface of the cylinder generated by the revolution of $BL$ about $RT = 2 \times \sqrt{r^2 - x^2} \times 2px$; $\therefore$ the numerator of the expression for $CO$ becomes $=$ the fluent of $4px^3\dot{x} \times \sqrt{r^2 - x^2}$, whose fluent (Fluent 75.) $= 4p \times \overline{-\frac{r^2y^3}{3} + \frac{1}{5}y^5}$ + the corr. Let $x=0$; then $y=r$; $\therefore$ the corr. of the fluent $= 4p \times \overline{\frac{2}{15}r^5 - \frac{1}{3}r^2y^3 + \frac{1}{5}y^5} =$ (when $x=r$, and $y=0$) $\frac{8pr^5}{15}$; and the content of the sphere $= \frac{4pr^3}{3}$; $\therefore$ $CO = \sqrt{\frac{\frac{8}{15}pr^5}{\frac{4pr^3}{3}}} = r \times \sqrt{\frac{2}{5}}$.

(74.) CASE 1.

Ex. 4. Let $SC$ be a straight line of uniform density suspended at one of its extremities $S$. To find the center of oscillation.



Let $O$ be the center of oscillation, and $SP$ a variable part $=x$. Then $SO = \frac{\text{the fluent of } x^2\dot{x}}{x \times \frac{x}{2}} = \frac{\text{the fluent of } x^2\dot{x}}{\frac{x^2}{2}}$ $= \frac{2}{3}x$, or for the whole line $= \frac{2SC}{3}$.

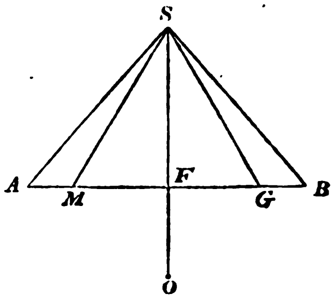
Ex. 5. To find the same, the density of the line being supposed to vary $\frac{1}{D^n}$ from the point of suspension.

In this case, $SO = \frac{\text{the fluent of } x^2\dot{x} \times \frac{1}{x^n}}{\text{the fluent of } \frac{x\dot{x}}{x^n}} = \frac{\text{the fluent of } x^{-n+2}\dot{x}}{\text{the fluent of } x^{-n+1}\dot{x}}$ $= \frac{2-n}{3-n} \times x$, for the whole line $= \frac{2-n}{3-n} \times SC$.

(75.) CASE 2.

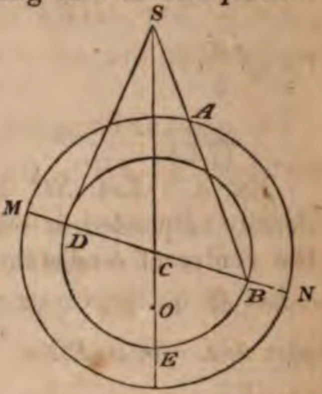
Ex. 6. Let $AB$ be a line vibrating lengthways in a vertical plane about $S$, having its two extremes $A$ and $B$ equidistant from the point of suspension. To find the center of oscillation.

Draw $SF$ perpendicular to $AB$; let $SF=a$, $FG=x$, and join $SG$. $F$ is the center of gravity of the line; then, taking $FM=FG$, we have $SO = \frac{2 \times \text{fluent of } \dot{x} \times \overline{a^2+x^2}}{2x \times a}$ $= \frac{a^2x + \frac{x^3}{3}}{ax} = a + \frac{x^2}{3a} = SF + \frac{FB^2}{3SF}$.



### (76.) CASE. 3.

Ex. 7. Let *ANM* be a circle vibrating in a vertical plane.



Let a diameter *MCN* cut a concentric circle in *D* and *B*. Assume $SC=a$, $CB=x$, $CN=r$. Then $SD^2+SB^2=2SC^2+2CB^2$; $\therefore$ the sum of the products of two particles at *D* and *B*, and the square of their distances from $S=\overline{a^2+x^2}\times$ the two particles; hence that expression for the whole circumference $=\overline{a^2+x^2}\times 2px=$ $2pa^2x+2px^3$; $\therefore SO=\dfrac{\text{fluent of } x^2\dot{s}}{\text{fluent of } x\dot{s}}=\dfrac{2p\times\text{fluent of }\overline{a^2x\dot{x}+x^3\dot{x}}}{p\times ax^2}$

$$=\frac{a^2x^2+\frac{x^4}{2}}{ax^2}=a+\frac{x^2}{2a}=\text{ for the whole circle } a+\frac{r^2}{2a}.$$

COR. If the point of suspension *S* be in the circumference of this circle, $SO=r+\frac{r}{2}$; $\therefore CO=\frac{r}{2}$; and conversely, if *O* be made the point of suspension, where $CO=\frac{r}{2}$, the center of oscillation will be in the circumference. (Art. 72. COR. 2.)

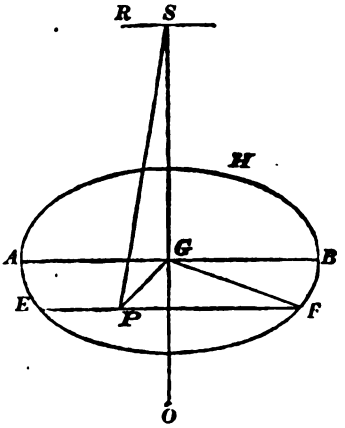
### (77.) CASE 4.

Ex. 8. Let *ANM* represent the circumference of a circle vibrating in a vertical plane.

By the last Case, the numerator of the expression for *SO*, or, as it is sometimes called, the force of the fluxion of the circle *BDE*, or of an annulus, whose breadth is $\dot{x}=2pa^2x\dot{x}+2px^3\dot{x}$; divide this quantity by $\dot{x}$, and there remains the force of the circumference $BDE=2pa^2x+2px^3$; hence, *SO* in this case $=\dfrac{2pa^2x+2px^3}{\text{the circumference } 2px\times a}=a+\dfrac{x^2}{a}$, or for the whole circle $=a+\dfrac{r^2}{a}$.

### (78.) CASE 5.

Ex. 9. Let *AHBE* be a circle, having its plane always perpendicular to the axis of suspension *SG*.



The point *G* being the center of the circle, let *AB* be the diameter parallel to the axis of motion *RS*; draw *EF* parallel to *AB*, *GP* perpendicular to *EF*, and join *SP*. Take $GB=r$, $GP=x$, $SG=a$; then $EF=2\sqrt{r^2-x^2}$, and $SO=$

$$\frac{\text{fluent of } 2\dot{x}\times\sqrt{r^2-x^2}\times\overline{a^2+x^2}}{\text{area of semi-circle}\times a}=$$

$$\frac{\text{the fluent of } 4a^2\dot{x}\sqrt{r^2-x^2}+4x^2\dot{x}\sqrt{r^2-x^2}}{\text{area of circle}\times a}= \text{(Fluent 24.)}$$

$$\frac{\overline{a^2+\frac{1}{4}r^2}\times pr^2}{pr^2\times a}=a+\frac{r^2}{4a}.$$

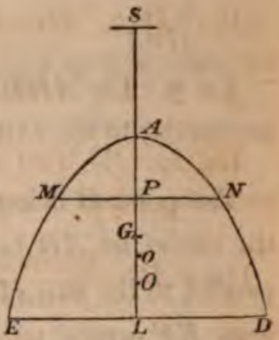
### (79.) CASE 6.

Ex. 10. Let *AHBE* represent the periphery of a circle, whose plane is always perpendicular to *SG*.

By the last Case, the numerator of the fraction, which expresses the value of *SO* for a circle, $=pa^2r^2+\frac{1}{4}pr^4$; of which the fluxion $2pa^2r\dot{r}+pr^3\dot{r}$ denotes the force of an annulus whose breadth is $\dot{r}$; ∴ dividing by $\dot{r}$, $2pa^2r+pr^3$ is the force of the periphery; hence, $SO=\dfrac{2pa^2r+pr^3}{2pr\times a}=a+\dfrac{r^2}{2a}$.

### (80.) CASE 7.

To find the center of oscillation of a curvilinear area moving flatways, or in a direction perpendicular to its axis.



Let $S$ be the point of suspension, $SA = d$, $AP = x$, $PN = y$; then $SO =$

$$\frac{\text{the fluent of } 2y\dot{x} \times \overline{d+x}|^2}{\text{the fluent of } 2y\dot{x} \times \dfrac{\text{fluent of } y\dot{x} \times \overline{d+x}}{\text{fluent of } y\dot{x}}}$$

$$= \frac{\text{the fluent of } y\dot{x} \times \overline{d+x}|^2}{\text{the fluent of } y\dot{x} \times \overline{d+x}}.$$

If $A$ be the point of suspension, $AO = \dfrac{\text{the fluent of } yx^2\dot{x}}{\text{the fluent of } yx\dot{x}}$.

To apply these expressions to any particular curve, the value of $y$ must be expressed in terms of $x$ from the nature of the curve, and the fluent can be found.

Ex. 11. Let the curve be the common parabola, and $S$ the point of suspension.

The equation is $y^2 = ax$; $\therefore SO = \dfrac{\text{fluent of } a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} \times \overline{d+x}|^2}{\text{fluent of } a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} \times \overline{d+x}}$

$$= \frac{\text{fluent of } a^{\frac{1}{2}} d^2 x^{\frac{1}{2}} \dot{x} + 2da^{\frac{1}{2}} x^{\frac{3}{2}} \dot{x} + a^{\frac{1}{2}} x^{\frac{5}{2}} \dot{x}}{\text{fluent of } a^{\frac{1}{2}} dx^{\frac{1}{2}} \dot{x} + a^{\frac{1}{2}} x^{\frac{3}{2}} \dot{x}} = \frac{\frac{2}{3} d^2 x^{\frac{3}{2}} + \frac{4}{5} d \times x^{\frac{5}{2}} + \frac{2}{7} x^{\frac{7}{2}}}{\frac{2}{3} dx^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}}}$$

$$= \frac{\frac{1}{3} d^2 + \frac{2}{5} dx + \frac{1}{7} x^2}{\frac{1}{3} d + \frac{1}{5} x}.$$

If $d = o$, or the point of suspension be $A$, $AO = \dfrac{5x}{7}$.

(81.) If $A$ be taken for the point of suspension, the value of $AO$ is determined by the expression $\dfrac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}}$; and since $AG$ can be found, $GO$ may be determined. Now $SG$ varies as $\dfrac{1}{GO}$; hence, if $o$ be the center of oscillation, corresponding to the point of suspension $S$, we have $AG : SG :: \dfrac{1}{GO} : \dfrac{1}{Go}$, from which $Go = \dfrac{AG \times GO}{SG}$; and therefore $So$ may be found.

If $SG=g$, $AG=a$, $AO=v$, and $GO=v-a$; then $Go=\frac{av-a^2}{g}$, and $So=g+\frac{av-a^2}{g}=g+\frac{a\times\overline{v-a}}{g}$.

Ex. 12. To find the center of oscillation of the common parabola, when $A$ is the point of suspension.

$$\text{Here } AO=\frac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}}=\frac{\text{fluent of } a^{\frac{1}{2}}x^{\frac{5}{2}}\dot{x}}{\text{fluent of } a^{\frac{1}{2}}x^{\frac{3}{2}}\dot{x}}=\frac{\frac{2}{7}x^{\frac{7}{2}}}{\frac{2}{5}x^{\frac{5}{2}}}=\frac{5x}{7}.$$

Now to determine $So$ for some other point of suspension $S$; let $SA=d$, $AO=\frac{5x}{7}=v$, $AG=\frac{3x}{5}=a$ (Art. 64, Ex. 2.); $\therefore\ GO=AO-AG=\frac{5x}{7}-\frac{3x}{5}$. Then, since $SG\times Go=AG\times GO$, or $Go=\frac{AG\times GO}{SG}=\frac{\frac{3x}{5}\times\overline{\frac{5x}{7}-\frac{3x}{5}}}{d+\frac{3x}{5}}$, we have $So$

$$=d+\frac{3x}{5}+\frac{\frac{3x}{5}\times\overline{\frac{5x}{7}-\frac{3x}{5}}}{d+\frac{3x}{5}},$$ which, reduced, gives the same result as in Ex. 11.

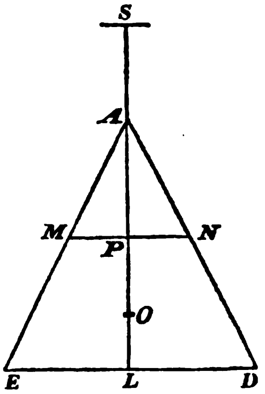
Ex. 13. Let $EAD$ represent any parabola, whose equation is $a^{n-1}y=x^n$, and the point of suspension $A$. (See the FIGURE in the preceding page.)

$$\text{Here } AO=\frac{\text{fluent of } yx^2\dot{x}}{\text{fluent of } yx\dot{x}}=\frac{\text{fluent of } x^{n+2}\dot{x}}{\text{fluent of } x^{n+1}\dot{x}}=\frac{n+2}{n+3}\times x.$$

If $n=\frac{1}{2}$, it becomes the common parabola; and $AO=\frac{5}{7}AL$ as before.

If $S$ be the point of suspension, $SO$ may be found from the expression $\frac{\text{fluent of } y\dot{x}\times\overline{d+x}|^2}{\text{fluent of } y\dot{x}\times\overline{d+x}}$ (Art. 80.) $=\frac{\text{fluent of } x^n\dot{x}\times\overline{d^2+2dx+x^2}}{\text{fluent of } x^n\dot{x}\times\overline{d+x}}$

$$=\frac{\overline{n+2}\,.\,\overline{n+3}\,.\,d^2+\overline{n+1}\,.\,\overline{n+3}\,.\,2dx+\overline{n+1}\,.\,\overline{n+2}\,.\,x^2}{\overline{n+2}\,.\,\overline{n+3}\,.\,d+\overline{n+1}\,.\,\overline{n+3}\,.\,x};$$ or by the method given in Art. 81.



Ex. 14. Let $AED$ be an isosceles triangle, suspended at $A$.

Take $AP = x$, $MN = y$, $AL = a$, $ED = b$; then $AO = \frac{\text{the fluent of } yx^2\dot{x}}{\text{the fluent of } yx\dot{x}} = \Big($since $y = \frac{bx}{a}\Big)$ $\frac{\text{the fluent of } x^3\dot{x}}{\text{the fluent of } x^2\dot{x}} = \frac{3x}{4} = \frac{3}{4}AL.$

If $S$ be the point of suspension, $So = \frac{\text{the fluent of } \overline{d+x}|^2 \times y\dot{x}}{\text{the fluent of } \overline{d+x}.y\dot{x}}$ $= \frac{6d^2 + 8dx + 3x^2}{6d + 4x}$; which, when $d = o$, becomes $\frac{3x}{4}$ as before.

(82.) CASE 8.

Let the proposed figure be an area $EAD$ vibrating edgeways, so that the motion of the axis is in the plane of the curve. (See the FIGURE in the opposite page.)

The sum of the products of each particle of the line $MN$, and the square of its distance from $S$, $SP^2 \times PN + \frac{PN^3}{3} =$ (Ex. 6.) $\overline{d+x}|^2 \times y + \frac{1}{3}y^3$; ∴ taking the area $AMN$, $SO = \frac{\text{the fluent of } \overline{d+x}|^2 \times y\dot{x} + \frac{1}{3}y^3\dot{x}}{\text{the fluent of } \overline{d+x}.y\dot{x}}$, whence, substituting for $y$ its value in any curve in terms of $x$, $SO$ may be found.

If $d = o$, or $A$ be the point of suspension, $AO = \frac{\text{the fluent of } yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{\text{the fluent of } yx\dot{x}}$.

Ex. 15. In the common parabola, suspended at $A$, $y^2 = ax$;

$$\therefore\ AO = \frac{\text{the fluent of } a^{\frac{1}{2}}x^{\frac{5}{2}}\dot{x} + \frac{1}{3}a^{\frac{3}{2}}x^{\frac{3}{2}}\dot{x}}{\text{the fluent of } a^{\frac{1}{2}}x^{\frac{3}{2}}\dot{x}} = \frac{\frac{2}{7}x^{\frac{7}{2}} + \frac{2}{15}ax^{\frac{5}{2}}}{\frac{2}{5}x^{\frac{5}{2}}} =$$

$\frac{5}{7}AL + \frac{1}{3}a$, for the whole area.

Ex. 16. In any parabola, whose equation is $a^{n-1}y = x^n$,

$$AO = \frac{\text{the fluent of } \frac{x^{n+2}\dot{x}}{a^{n-2}} + \frac{1}{3}\frac{x^{3n}\dot{x}}{a^{3n-3}}}{\text{the fluent of } \frac{x^{n+1}\dot{x}}{a^{n-1}}} = \frac{\frac{x^{n+3}}{n+3} + \frac{1}{9n+3} \times \frac{x^{3n+1}}{a^{2n-2}}}{\frac{x^{n+2}}{n+2}} =$$

$$\frac{n+2}{n+3} \times x + \frac{n+2}{9n+3} \times \frac{x^{2n-1}}{a^{2n-2}}.$$

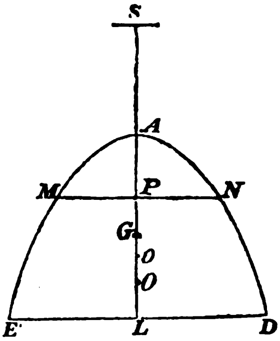
Ex. 17. In an isosceles triangle, $y : x :: b : a$; $\therefore AO =$

$$\frac{\text{the fluent of } \frac{b}{a}x^3\dot{x} + \frac{1}{3} \times \frac{b^3}{a^3}x^3\dot{x}}{\text{the fluent of } \frac{b}{a}x^2\dot{x}} = \frac{3}{4}x + \frac{1}{4}\frac{b^2x}{a^2},$$ when $x = a$, and

$y = b$, $= \frac{3}{4}a + \frac{b^2}{4a}$.

(83.) CASE 9.

Let $EAD$ represent a curve line vibrating flatways.



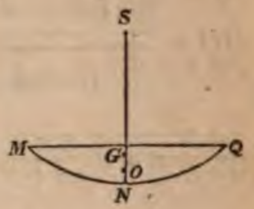
Since each particle of the line $MN$ moves as with a radius $SP$, if the fluxion of $AM = \dot{z}$, $SO = \dfrac{\text{fluent of } \overline{d+x}|^2 \times \dot{z}}{\text{fluent of } \overline{d+x} \times \dot{z}}$.

Hence, if $z$ can be expressed in terms of $x$, $SO$ can be found. If $A$ be the point of suspension, $d = o$, and $AO = \dfrac{\text{fluent of } x^2\dot{z}}{\text{fluent of } x\dot{z}}$.

Ex. 18. Let $AED$ be an isosceles triangle; $s$ one side, and

$a$ the perpendicular, the point of suspension being $A$. Then $\dot{z} = \frac{s\dot{x}}{a}$; and $AO = \frac{\text{the fluent of } x^2\dot{x}}{\text{the fluent of } x\dot{x}} = \frac{2}{3}AL$.

Ex. 19. To find the center of oscillation of any given arc of a circle $MNQ$, the point of suspension being in the center of the circle.



Let $G$ be the center of gravity of the arc, $MQ$ its chord. Then $SO = \frac{SN^2 \times \text{arc } MNQ}{SG \times \text{arc } MNQ}$ $= \frac{SN^2}{SG} = \frac{SN^2}{SN \times \frac{MQ}{MNQ}}$, (for $SG = \frac{SN \times MQ}{MNQ}$, Art. 64, Ex. 12.) $= \frac{SN \times MNQ}{MQ}$.

(84.) Case 10.

Let $EAD$ be a curve line vibrating edgeways.

If $S$ be the point of suspension, the radius of the circle in which $\dot{z}$ moves $= \sqrt{\overline{d+x}|^2 + y^2}$; $\therefore SO = \frac{\text{fluent of } \dot{z} \times \overline{d+x}|^2 + y^2\dot{z}}{\text{fluent of } \overline{d+x} \times \dot{z}}$.

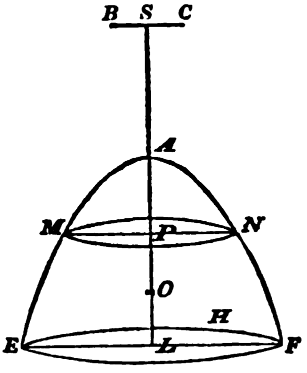
If $A$ be the point of suspension, $AO = \frac{\text{fluent of } x^2\dot{z} + y^2\dot{z}}{\text{fluent of } x\dot{z}}$.

Ex. 20. Let the figure be an isosceles triangle, whose altitude is $a$, and $b = \frac{1}{2}$ the base. Then $\dot{z} = \frac{s\dot{x}}{a}$; $y^2 = \frac{b^2x^2}{a^2}$; $\therefore AO =$

$$\frac{\text{fluent of } x^2 \times \frac{s\dot{x}}{a} + \frac{b^2x^2}{a^2} \times \frac{s\dot{x}}{a}}{\text{fluent of } \frac{sx\dot{x}}{a}} = \frac{\frac{1}{3}x^3 + \frac{b^2x^3}{3a^2}}{\frac{x^2}{2}} = (\text{if } x = a)\ \frac{2}{3}a + \frac{2b^2}{3a}.$$

(85.) Case 11.

Let the figure $AEF$ be a solid, generated by the rotation of the surface $EAF$ about its axis $AL$, having its base $HH$ parallel to the axis of motion $BC$.



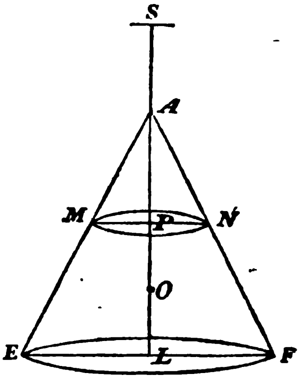
Take the circle *MN*, whose plane is parallel to *HH*. It appears, by Ex. 9, that the sum of the products of each particle of the circle *MN*, and the square of its distance from the axis, $=\overline{SP^2+\frac{1}{4}PN^2}\times$ the circle $MN=\overline{\overline{d+x}|^2+\frac{1}{4}y^2}\times py^2$; hence, $py^2\dot{x}\times\overline{\overline{d+x}|^2+\frac{1}{4}y^2}$ is the fluxion of the sum of these products; and $SO=\dfrac{\text{fluent of } py^2\times\overline{d+x}|^2\dot{x}+\frac{1}{4}py^4\dot{x}}{\text{fluent of } py^2\dot{x}\times\overline{d+x}}$

$$=\frac{\text{the fluent of } y^2\times\overline{d+x}|^2\dot{x}+\frac{1}{4}y^4\dot{x}}{\text{the fluent of } y^2\dot{x}\times\overline{d+x}}.$$

If the point of suspension be at the vertex *A*, $d=o$, and

$$AO=\frac{\text{the fluent of } y^2x^2\dot{x}+\frac{1}{4}y^4\dot{x}}{\text{the fluent of } y^2x\dot{x}}.$$

Ex. 21. Let the solid be a cone, and *S* the point of suspension.



Take $AL=a$, $LF=b$; then $a:b::x:y$; $\therefore y=\frac{bx}{a}=mx$, if $\frac{b}{a}=m$. Hence, the fluxion of the numerator, which $=y^2\dot{x}\times\overline{d+x}|^2+\frac{1}{4}y^4\dot{x}$, becomes $m^2x^2\dot{x}\times\overline{d^2+2dx+x^2}+\frac{1}{4}m^4x^4\dot{x}$, and the fluent $=\frac{m^2d^2x^3}{3}+\frac{m^2dx^4}{2}+\frac{m^2x^5}{5}+\frac{m^4x^5}{20}$. Also the fluent of the denominator $y^2\dot{x}\times\overline{d+x}=$ the fluent of $m^2dx^2\dot{x}+m^2x^3\dot{x}=\frac{m^2dx^3}{3}+\frac{m^2x^4}{4}$; $\therefore SO=$

$$\frac{\frac{1}{3}m^2d^2x^3+\frac{1}{2}m^2dx^4+\frac{1}{5}m^2x^5+\frac{1}{20}m^4x^5}{\frac{1}{3}m^2dx^3+\frac{1}{4}m^2x^4}=\frac{20d^2+30dx+12x^2+3m^2x^2}{20d+15x}$$

by multiplying the whole of the numerator and denominator by 60, and dividing by $m^2x^3$; which for the whole cone $= \frac{20d^2+30da+12a^2+3b^2}{20d+15a}$.

Cor. 1. If $d=o$, or $A$ be the point of suspension, $AO=\frac{4a^2+b^2}{5a}$.

Cor. 2. If a cone be suspended by the vertex, and the center of oscillation be in the base, $a=b$. For in this case, $\frac{4a^2+b^2}{5a}=a$; $\therefore 4a^2+b^2=5a^2$, and $a=b$, a property of the right cone.

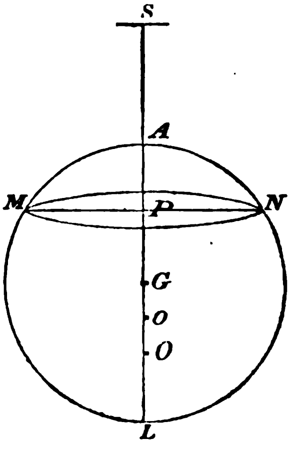
Ex. 22. Let $AEF$ be a paraboloid, suspended at $S$.

Since $y^2=ax$, $SO$ in this case $=\frac{\text{fluent of } ax\dot{x}\times\overline{d^2+2dx+x^2}+\frac{1}{4}a^2x^2\dot{x}}{\text{fluent of } ax\dot{x}\times\overline{d+x}}$

$$=\frac{\frac{1}{2}ad^2x^2+\frac{2}{3}adx^3+\frac{1}{4}ax^4+\frac{1}{12}a^2x^3}{\frac{1}{2}adx^2+\frac{1}{3}ax^3}=\frac{6d^2+8dx+3x^2+ax}{6d+4x}.$$

If $d=o$, $AO=\frac{3x+a}{4}$.

Ex. 23. Let the solid be a sphere, to find $AO$ and $So$.



In this case it is most convenient to determine the value of $AO$, and to deduce $SO$ from it by Art. 70.

Since $y^2=2rx-x^2$, $AO=p\times$ fluent of

$$\frac{\overline{2rx-x^2}\times x^2\dot{x}+\frac{1}{4}\times\overline{4r^2x^2-4rx^3+x^4}\times\dot{x}}{\text{sphere}\times AG}$$

$$=p\times\text{fluent of }\frac{r^2x^2\dot{x}+rx^3\dot{x}-\frac{3}{4}x^4\dot{x}}{\text{sphere}\times AG}=$$

$$p\times:\frac{\frac{1}{3}r^2x^3+\frac{1}{4}rx^4-\frac{3}{20}x^5}{\frac{4pr^3}{3}\times r}=(\text{when } x=2r)\ \frac{\frac{28}{15}r^5}{\frac{4}{3}r^4}=\frac{7r}{5}.$$

Hence, $GO=\frac{2r}{5}$; $\therefore$ if $o$ be the center of oscillation, corresponding to the point of suspension $S$, since $SG\times Go=AG\times GO$ (Art. 72.); $\therefore\ \overline{d+r}\times Go=r\times\frac{2r}{5}=\frac{2r^2}{5}$; $\therefore\ Go=\frac{2r^2}{5\times\overline{d+r}}$; $\therefore\ So=d+r+\frac{2r^2}{5\times\overline{d+r}}$.

If $SA=r$, $So=2r+\frac{2r^2}{10r}=2r+\frac{r}{5}=\frac{11r}{5}$.

Ex. 24. Let the solid be a cylinder, and $S$ the point of suspension.

Here $y$ is constant $=b$; $\therefore\ SO=\frac{b^2\dot{x}\times\overline{d^2+2dx+x^2}+\frac{1}{4}b^4\dot{x}}{b^2\dot{x}\times\overline{d+x}}$

$$=\frac{\text{fluent of } d^2\dot{x}+2dx\dot{x}+x^2\dot{x}+\frac{1}{4}b^2\dot{x}}{\text{fluent of } d\dot{x}+x\dot{x}}=\frac{d^2x+dx^2+\frac{1}{3}x^3+\frac{1}{4}b^2x}{dx+\frac{1}{2}x^2}.$$

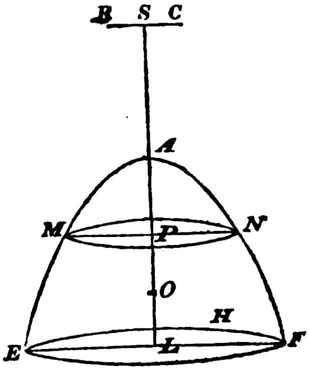
Let $d=o$; $AO=\frac{\frac{1}{3}x^2+\frac{1}{4}b^2}{\frac{1}{2}x}=\frac{2x}{3}+\frac{b^2}{2x}$.

(86.) CASE 12.

Let $EAF$ be the superficies of a solid generated by the revolution of $EAF$ round its axis; to find the center of oscillation of the superficies.

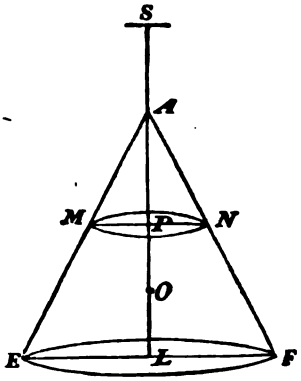
By Ex. 10. the force of the periphery of a circle $MN=2pr\times\overline{a^2+\frac{1}{2}r^2}$ $=2py\times\overline{\overline{d+x}|^2+\frac{1}{2}y^2}$; therefore the fluxion of the force of the superficies $=2py\dot{z}\times\overline{\overline{d+x}|^2+\frac{1}{2}y^2}$;

$$\therefore\ SO=\frac{f.\ 2py\dot{z}\times\overline{\overline{d+x}|^2+\frac{1}{2}y^2}}{\text{fluent of } 2py\dot{z}\times\overline{d+x}}.$$



If $d=o$, $AO=\dfrac{\text{the fluent of } yx^2\dot{z}+\frac{1}{2}y^3\dot{z}}{\text{the fluent of } xy\dot{z}}$.

Ex. 25. Let $EAF$ be a cone.



If $AE=s$, $AL=a$, $EL=b$, $AM=z$, $\dot{z} : \dot{x} :: s : a$; $\therefore \dot{z}=\dfrac{s\dot{x}}{a}=m\dot{x}$ if $\dfrac{s}{a}=m$; and $x : y :: a : b$; $\therefore y=\dfrac{bx}{a}=nx$, if $\dfrac{b}{a}=n$; hence $SO=$

$$\frac{f.\ nx\times m\dot{x}\times \overline{d^2+2dx+x^2+\frac{1}{2}n^2x^2}}{f.\ nx\times m\dot{x}\times \overline{d+x}}=$$

$$\frac{f.\ d^2x\dot{x}+2dx^2\dot{x}+x^3\dot{x}+\frac{1}{2}n^2x^3\dot{x}}{f.\ dx\dot{x}+x^2\dot{x}}=\frac{\frac{1}{2}d^2x^2+\frac{2}{3}dx^3+\frac{1}{4}x^4+\frac{1}{8}n^2x^4}{\frac{1}{2}dx^2+\frac{1}{3}x^3}$$

$=$ for the whole cone, $\dfrac{12d^2+16da+6a^2+3b^2}{12d+8a}$.

Let $d=o$; then $AO=\dfrac{6a^2+3b^2}{8a}$.

Ex. 26. Let $EAF$ be the superficies of a sphere.

Here $\dot{z} : \dot{x} :: r : y$; $\therefore y\dot{z}=r\dot{x}$. Hence, if $A$ be the point of suspension, $AO=\dfrac{\text{fluent of } rx^2\dot{x}+\frac{1}{2}ry^2\dot{x}}{\text{fluent of } rx\dot{x}}$; since $y^2=2rx-x^2$

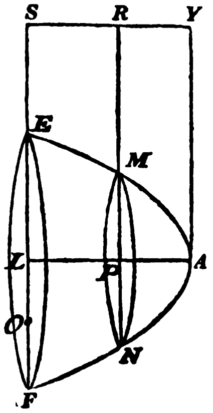
$$=\frac{\text{fluent of } rx^2\dot{x}+r^2x\dot{x}-\frac{1}{2}rx^2\dot{x}}{\text{fluent of } rx\dot{x}}=\frac{\text{fluent of } \frac{1}{2}rx^2\dot{x}+r^2x\dot{x}}{\text{fluent of } rx\dot{x}}=$$

$\dfrac{\frac{1}{6}rx^3+\frac{1}{2}r^2x^2}{\frac{rx^2}{2}}=$ (if $x=2r$) for the whole sphere $\dfrac{5r}{3}$.

(87.) CASE 13.

Let the figure *EAF* be a solid, but its axis parallel to the axis of motion [illegible]*RS*.

y



Take $RP=a$, $AP=x$, $PM=y$; then, by Ex. 7, the force of the particles in the circle $MN=\overline{a^2+\frac{1}{2}y^2}\times py^2$; $\therefore$ $SO$ for this solid

$$=\frac{\text{the fluent of }\overline{a^2+\frac{1}{2}y^2}\times py^2\dot{x}}{\text{solid}\times a}=$$

$$\frac{\text{the fluent of }py^2\dot{x}\times a^2+\frac{1}{2}py^4\dot{x}}{\text{the fluent of }py^2\dot{x}\times a}=a+$$

$$\frac{\text{the fluent of }\frac{1}{2}y^4\dot{x}}{a\times\text{the fluent of }y^2\dot{x}}.$$

Ex. 27. Let *EAF* be the segment of a sphere, of radius $r$. Let $y^2=2rx-x^2$, and $y^4=4r^2x^2-4rx^3+x^4$; $\therefore$ $SO=a+$

$$\frac{\text{fluent of }2r^2x^2\dot{x}-2rx^3\dot{x}+\frac{1}{2}x^4\dot{x}}{a\times\text{fluent of }2rx\dot{x}-x^2\dot{x}}=a+\frac{\frac{2}{3}r^2x-\frac{1}{2}rx^2+\frac{1}{10}x^3}{a\times\overline{r-\frac{1}{3}x}}.$$

If $x=r$, then $So$ for the hemisphere $=a+\frac{2r^2}{5a}$; and the same expression is true for the sphere.

Ex. 28. Let *EAF* be a paraboloid, whose equation is $y^2=cx$.

$$SO=a+\frac{\text{the fluent of }\frac{1}{2}c^2x^2\dot{x}}{a\times\text{the fluent of }cx\dot{x}}=a+\frac{\frac{1}{6}c^2x^3}{\frac{1}{2}acx^2}=a+\frac{cx}{3a}=$$

$$a+\frac{y^2}{3a}.$$

Ex. 29. Let *EAF* be any paraboloid, whose equation is $c^{n-1}y = x^n$.

Here $y^2 = \frac{x^{2n}}{c^{2n-2}}$, and $y^4 = \frac{x^{4n}}{c^{4n-4}}$; $\therefore SO = a +$

$$\frac{f.\,\frac{1}{2}\frac{x^{4n}\dot{x}}{c^{4n-4}}}{a\times f.\,\frac{x^{2n}\dot{x}}{c^{2n-2}}} = a + \frac{\frac{1}{8n+2}\times\frac{x^{4n+1}}{c^{4n-4}}}{\frac{a}{2n+1}\times\frac{x^{2n+1}}{c^{2n-2}}} = a + \frac{2n+1}{8n+2}\times\frac{x^{2n}}{ac^{2n-2}} = a +$$

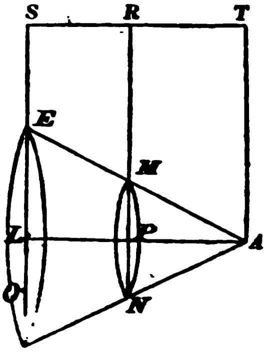
$$\frac{2n+1}{\overline{8n+2}.a}\times y^2.$$

If $n = \frac{1}{2}$, it is the common parabola; and $SO = a + \frac{y^2}{3a}$, as before.

Ex. 30. If the figure be a cone. Let $AL = c$, $LE = b$; $y : x :: b : c$; $\therefore y = \frac{bx}{c} = mx$ if $m = \frac{b}{c}$; $\therefore SO = a +$

$$\frac{f.\,\frac{1}{2}m^4x^4\dot{x}}{a\times f.\,m^2x^2\dot{x}} = a + \frac{\frac{1}{10}m^4x^5}{\frac{1}{3}a} = a + \frac{3}{10}\times$$

$\frac{b^2}{a}$ for the whole cone.



Ex. 31. In the cylinder $y$ is constant, and $SO = a + \frac{y^2}{2a}$.

(88.) CASE. 14.

Let *EAF* represent a superficies, with its axis parallel to the axis of motion. (See the FIGURE in p. 83.)

By Ex. 8. the force of the circumference of a circle vibrating in this manner, $= 2pa^2y + 2py^3$; $\therefore$ the fluxion of the force of this superficies $= \overline{2pa^2y + 2py^3}\times\dot{x}$; $\therefore SO = \frac{f.\,2pa^2y\dot{x} + 2py^3\dot{x}}{a\times f.\,2py\dot{x}}$

$=a+\frac{f.\,y^3\dot{z}}{a\times f.\,y\dot{z}}$; whence, by substituting for $\dot{z}$ in terms of $y$, $SO$ may be found.

Ex. 32. Let *EAF* be a spherical superficies.

If $\dot{x}$ = the versed sine, and $y$ the ordinate, $\dot{z}=\frac{r\dot{x}}{y}$; $\therefore$ $SO=$

$$a+\frac{f.\,\frac{y^3\times r\dot{x}}{y}}{a\times f.\,r\dot{x}}=a+\frac{f.\,\overline{2rx-x^2}\times\dot{x}}{a\times f.\,\dot{x}}=a+\frac{rx-\frac{1}{3}x^2}{a}.$$

If $x=r$, or $2r$, $SO=a+\frac{2r^2}{3a}$.

# CHAP. XII.

## ON SECOND, THIRD, &c. FLUXIONS.

(89.) THE fluxion of a variable quantity has been considered as its rate of increase or decrease; hence, if that increase or decrease be uniform, the fluxion continues the same. But if the rate of increase or decrease be variable, its measure will also be variable; and will itself have a certain rate of increase or decrease. The measure of this rate will be its fluxion; that is, the fluxion of the fluxion, or the second fluxion of the variable quantity. If this second fluxion be also variable, the measure of its rate of variation will be the third fluxion of the original quantity; and so on, till some fluxion becomes constant; then it will have no more. These different orders of fluxions, it is plain, are similar in their nature to the first fluxions; for they are such, in fact, to the quantities from which they are deduced; and their fluents are the fluxions which immediately precede them. The first fluxion of $x$ being denoted by $\dot{x}$, the second fluxion is denoted by $\ddot{x}$, the third by $\dddot{x}$, &c.

### (90.) EXAMPLES.

Ex. 1. The fluxion of $x^2$ is $2x\dot{x}$; if $x$ increase with an uniform velocity, $\dot{x}$ is constant; but $x$ being variable, $2x$ admits a fluxion $2\dot{x}$; and the second fluxion is $2\dot{x}^2$. But if $x$ do not increase uniformly, $\dot{x}$ is not always the same; hence, it admits a fluxion as $\ddot{x}$; so that the second fluxion of $x^2$ is $2x\ddot{x}+2\dot{x}^2$. If $\ddot{x}$ be variable, we have a third fluxion, $2\dot{x}\ddot{x}+2x\dddot{x}+4\dot{x}\ddot{x}=2x\dddot{x}+6\dot{x}\ddot{x}$. Should $\dddot{x}$ be variable, it admits a fourth; $2\dot{x}\dddot{x}+2x\ddddot{x}+6\ddot{x}^2+6\dot{x}\dddot{x}$; and thus we may proceed till some one fluxion is constant.

(91.) A simple quantity $x^n$, where $n$ is an affirmative whole number, has $n$ fluxions, if $\dot{x}$ be constant.

For the first fluxion is $nx^{n-1}\dot{x}$; and $\dot{x}$ being constant;

the second - - - - - $n.\overline{n-1}.x^{n-2}\dot{x}^2$;

the third - - - - - $n.\overline{n-1}.\overline{n-2}.x^{n-3}\dot{x}^3$;

the $n^{th}$ - - - - - $n.\overline{n-1}.....\overline{n-n+1}.x^0\dot{x}^n$, which is constant.

Ex. 2. To find the third fluxion of $ax^4+by^3$.

The first fluxion $= 4ax^3\dot{x}+3by^2\dot{y}$, where $x$, $\dot{x}$, $y$, and $\dot{y}$, are all, by hypothesis, variable; ∴ the second fluxion $= 12ax^2\dot{x}^2+4ax^3\ddot{x}+6by\dot{y}^2+3by^2\ddot{y}$; and for the same reason the third fluxion $= 24ax\dot{x}^3 + 24ax^2\dot{x}\ddot{x}+12ax^2\dot{x}\ddot{x}+4ax^3\dddot{x}+6b\dot{y}^3 +12by\dot{y}\ddot{y}+6by\dot{y}\ddot{y}+3by^2\dddot{y} = 24ax\dot{x}^3+36ax^2\dot{x}\ddot{x}+4ax^3\dddot{x}+6b\dot{y}^3+ 18by\dot{y}\ddot{y}+3by^2\dddot{y}$.

Ex. 3. To find the second fluxion of $x^3y^2$.

The first fluxion $= 3x^2\dot{x}y^2+2x^3y\dot{y}$; hence, the second fluxion $=6x\dot{x}^2y^2+6x^2\dot{x}y\dot{y}+3x^2y^2\ddot{x}+6x^2\dot{x}y\dot{y}+2x^3\dot{y}^2+2x^3y\ddot{y}=6x\dot{x}^2y^2 +12x^2\dot{x}y\dot{y}+3x^2y^2\ddot{x}+2x^3\dot{y}^2+2x^3y\ddot{y}$.

Ex. 4. To find the second fluxion of $x^m y^n$.

The first fluxion $= mx^{m-1}\dot{x}y^n+ny^{n-1}\dot{y}x^m$; and the second fluxion $=m\,,\overline{m-1}.x^{m-2}\dot{x}^2y^n+mx^{m-1}\ddot{x}y^n+mnx^{m-1}\dot{x}y^{n-1}\dot{y}+n.\overline{n-1}.y^{n-2} \dot{y}^2x^m+ny^{n-1}\ddot{y}x^m+mny^{n-1}\dot{y}x^{m-1}\dot{x}$.

# CHAP. XIII.

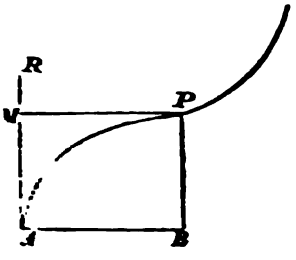
## ON THE POINT OF CONTRARY FLEXURE OF A CURVE.

DEF. If a curve be in one part concave and in another convex to its axis, the point where it changes from concave to convex, or from convex to concave is the point of contrary flexure.

(92.) The abscissa being supposed to flow uniformly, it appears by COR. Art. 30, that whilst the fluxion of the ordinate decreases the curve is concave, when it increases convex to the axis; therefore in the limit between the two, at the extremity of the concave part, the fluxion of the ordinate neither increases nor decreases, and here it is a minimum; therefore its fluxion or $-\ddot{y}=0$.

In the same manner if the curve changes from convex to concave, at the limiting point, the fluxion of $y$ is a maximum; or $+\ddot{y}=0$.

We have here considered $AB$ the abscissa, and $PB$ the ordinate. From $A$, draw $AM$ perpendicular to $AB$, and let $AM$ be considered as the abscissa, and $MP$ the ordinate to $AR$ as an axis; then at $P$, the point of contrary flexure $\ddot{x}=0$; its sign being positive, when the curve is convex to the axis, and negative, when it is concave.



(93.) Hence the RULE. To determine the point of contrary flexure, take the fluxion of the equation; suppose $\dot{x}$ or $\dot{y}$

constant; take the fluxion again; make $\ddot{y}$ or $\ddot{x}=0$; and the value of $x$ or $y$ may be found.

And to determine, whether the part of the curve between two given points be convex or concave to the axis, observe whether the value of the expression for $\ddot{y}$ be positive or negative; if positive, the curve is convex; if negative, concave; the whole curve being supposed to be on one side of the axis.

### (94.) EXAMPLES.

Ex. 1. Let the equation to the curve be $y=ax+bx^2-cx^3$.

$$\begin{aligned} \text{Here } \dot{y} &= a\dot{x} + 2bx\dot{x} - 3cx^2\dot{x}, \text{ if } \dot{x}=1; \\ &= a + 2bx - 3cx^2; \\ \therefore \ddot{y} &= 2b\dot{x} - 6cx\dot{x}; \text{ make } \ddot{y}=0; \end{aligned}$$

then, $6cx = 2b$, and $x = \dfrac{2b}{6c} = \dfrac{b}{3c} =$ the value of $x$ at the point of contrary flexure.

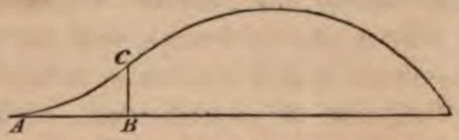
Ex. 2. Let the equation be $y = x + 36x^2 + 2x^3 - x^4$.

$$\begin{aligned} \dot{y} &= \dot{x} + 72x\dot{x} + 6x^2\dot{x} - 4x^3\dot{x}, \text{ if } \dot{x}=1; \\ &= 1 + 72x + 6x^2 - 4x^3; \\ \therefore \ddot{y} &= 72\dot{x} + 12x\dot{x} - 12x^2\dot{x}; \text{ make } \ddot{y}=0; \\ \therefore 0 &= 72 + 12x - 12x^2; \end{aligned}$$

hence, $x^2 - x = 6$;

$\therefore\ x^2 - x + \frac{1}{4} = \dfrac{25}{4}$, and $x = 3$, the positive value.

If $x$ be assumed less than 3 in the equation $72+12x-12x^2$ which is the value of $\ddot{y}$, the result is positive; if greater than 3, the result is negative; if therefore $AB=3$, and $BC$ be drawn perpendicular to $AB$, from $A$ to $C$ the curve is convex to the axis, and afterwards concave.



Ex. 3. Let $y=x^4-12x^3+48x^2-64x$, to find the point of contrary flexure.

Here $\dot{y} = 4x^3 - 36x^2 + 96x - 64$, if $\dot{x} = 1$;
$\therefore\ \ddot{y} = 12x^2 - 72x + 96$;
Let $\ddot{y} = 0$; then $x^2 - 6x + 8 = 0$, or $\overline{x-2}\,.\,\overline{x-4} = 0$;
$\therefore\ \ddot{y} = 0$, when $x = 2$, and 4.

If $x$ be less than 2, $\ddot{y}$ is positive, or the curve is convex to its axis; if $x$ be greater than 2, and less than 4, $\ddot{y}$ is negative, and the curve is concave; if $x$ be greater than 4, convex.

Ex. 4. In general, let the equation be $a^4y = 3x^5 - 35ax^4 + 140a^2x^3 - 240a^3x^2$; $a^4\dot{y} = 15x^4 - 140ax^3 + 420a^2x^2 - 480a^3x$, if $\dot{x} = 1$; $\therefore\ a^4\ddot{y} = 60x^3 - 420ax^2 + 840a^2x - 480a^3$;

Let $\ddot{y} = 0$; then this expression $= 0$; or, dividing by 60,
$0 = x^3 - 7ax^2 + 14a^2x - 8a^3$;
or, $0 = \overline{x-a} \times \overline{x-2a} \times \overline{x-4a}$; therefore the curve has three points of contrary flexure corresponding to the value of $x$, as it $= a$, or $2a$, or $4a$.

Let $x$ be less than $a$; $\ddot{y}$ is negative, or the curve concave.

If $x$ be greater than $a$ but less than $2a$, $\ddot{y}$ is positive, or the curve convex.

If $x$ be greater than $2a$ but less than $4a$, the curve is concave.

(95.) If the equation which expresses the value of $\ddot{y}$ have two equal roots, then $\ddot{y}$ does not change its sign in passing through 0; therefore the point determined by assuming $\ddot{y} = 0$, is not a point of contrary flexure. This will happen when the equation has an even number of equal roots. (Art. 27.)

# CHAP. XIV.

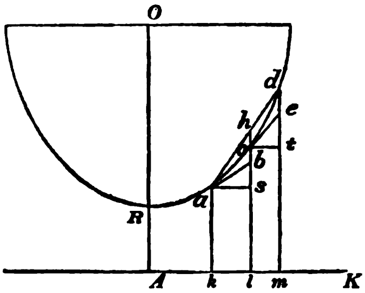
## ON THE RADIUS OF CURVATURE.

### SECTION I.

### PROPOSITION.

(96.) IN the common parabola, if the abscissa be perpendicular, and the ordinate parallel to the axis, the part of the subsequent ordinate intercepted between the curve and the tangent, or the deflection from the tangent, is equal to half the second fluxion of the ordinate.

Let *AK* be the abscissa, and *ARO* the axis of the parabola; *ak*, *lo*, *md*, three equidistant ordinates; join *ad*, and at *a* and *o*, draw tangents *ab*, *oe*; draw *as*, *ot*, parallel to *Am*; then *as*, *ot*, may represent the fluxions of the abscissa, and *sb*, *te*, those of the ordinate. Now by the nature of the parabola, *lh* bisects *ad*, and *bo* = *oh*; also *oe* is parallel to *hd*; therefore *ho* = *de*, and consequently *bo*=*de*; or the deflections from the tangent are uniform. But these deflections, *bo* and *de*, are produced by accelerations of velocity, which were nothing at *a* and *o*; therefore 2*bo* and 2*de* will represent the spaces that would be described by the uniform rate of increase, whilst *bo* or *de* is described, or they represent the fluxions of *sb* and *te*; that is, *bo* or *de* = $\frac{1}{2}$ the fluxion of $\dot{y}$ or $= \frac{1}{2}\ddot{y}$.



COR. If *bo*, and *de*, be considered as produced by accelerating forces, which were nothing at *a* and *o*, and act in the directions of the ordinates, those accelerating forces must in the parabola be uniform.

### (97.) PROPOSITION.

In any algebraic curve, the deflexion from the tangent is ultimately equal to half the second fluxion of the ordinate.

In the parabola this accurately obtains, and will obtain in all curves, where the accelerating force, as in the last Corollary, can be considered as constant. By Art. 7., if the time be indefinitely diminished, the space described by a variable velocity which was nothing at first, vanishes in respect of the space described by an uniform velocity; and in the same manner the deflection from the tangent caused by a variable force, which was nothing at *a*, vanishes in respect of *bo* described by the constant force, when the time is diminished in infinitum; that is, in any algebraic curve, the deflection from the tangent ultimately equals half the second fluxion of the ordinate.

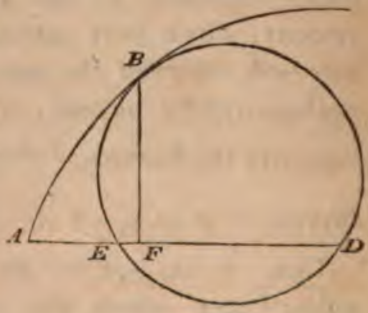
### (98.) PROPOSITION.

If two algebraic curves have the same fluxions of the abscissa and ordinate, they will have the same tangent; and if the second fluxion of the ordinate be the same in both cases, they will have the same curvature.

The first part of this Proposition is evident from Art. 30., and the second is immediately deducible from the last Proposition.

### (99.) DEFINITION.

Let *AB* be any curve; *BED* a circle, touching the curve in *B*; draw the ordinate *BF*; then *AF* flowing uniformly, if the second fluxion of the ordinate *BF* be the same both in the circle and the curve, *BED* is called the Circle of Curvature.



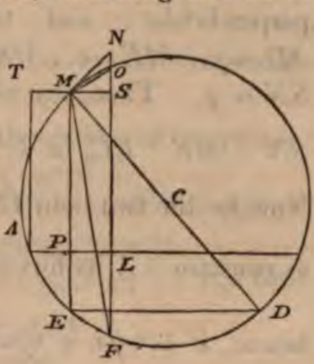
### (100.) PROPOSITION.

To find the radius of curvature in any algebraic curve, in terms of the fluxions of the arc, of the ordinate, and the abscissa; the fluxion of the abscissa being constant.

Let *MAD* be the circle of curvature, touching the curve at *M*; *AP* the abscissa; *MP*, *NL*, two ordinates; *C* the center, and *MCD* the diameter of the circle; produce *MP* to *E*, and join *DE*. Let $AP = x$, $MP = y$, $MN = \dot{z}$, $SN = \dot{y}$, and $ON = \frac{1}{2}\ddot{y}$; join *MO*, *MF*. Then, by similar triangles, *NOM*, *NMF*, $NO : NM :: MN : NF$ or (Art. 97.) $-\frac{1}{2}\ddot{y} : \dot{z} :: \dot{z} :$ $NF = \dfrac{\dot{z}^2}{-\frac{1}{2}\ddot{y}}$. Now when *O* and *M* nearly coincide, *NF* ultimately $=$ *ME*; and, by similar triangles, *NMS*, *DME*,

$$NM : MS :: DM : ME$$

or $\dot{z} : \dot{x} :: DM : \dfrac{\dot{z}^2}{-\frac{1}{2}\ddot{y}}$; $\therefore DM = \dfrac{-\dot{z}^3}{\frac{1}{2}\dot{x}\ddot{y}}$; and *CM* the radius of curvature $= \dfrac{-\dot{z}^3}{\dot{x}\ddot{y}}$; $\dot{x}$ being constant.



(101.) Nearly in the same manner, by taking *AT* as the abscissa perpendicular to *AP*, and *TM* as the ordinate, the radius of curvature $= \dfrac{\dot{z}^3}{\dot{y}\ddot{x}}$; for $\dot{y}$ is uniform by supposition, and the second fluxion of $x$ is positive, on account of the convexity of the curve.

(102.) In general, when $\dot{x}$ and $\dot{y}$ are both variable, the radius of curvature $= \dfrac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$.

$\dot{x}=1$. The radius of curvature in this case $= \dfrac{\dot{z}^3}{-\ddot{y}}$.

Now, by similar triangles, *MSN*, *BOT*, *DTO*; $\dot{z} : \dot{x} :: BO : OT :: DO : DT :: \sqrt{ay} : y :: \sqrt{a} : \sqrt{y}$; $\therefore \dot{z} = \overline{\dfrac{a}{y}}\Big|^{\frac{1}{2}}$ and $\dot{z}^3 = \dfrac{a^{\frac{3}{2}}}{y^{\frac{3}{2}}}$. Again, by the same triangles, $\dot{x} : \dot{y} :: OT : TB :: DT : TO :: y : \sqrt{ay-y^2} :: y^{\frac{1}{2}} : \sqrt{a-y}$; $\therefore \dot{y} = \sqrt{\dfrac{a-y}{y}}$; and $\ddot{y} =$ the fluxion of $\overline{a-y}|^{\frac{1}{2}} \times y^{-\frac{1}{2}} = \dfrac{-\frac{1}{2}\dot{y}}{y^{\frac{1}{2}} \times \overline{a-y}|^{\frac{1}{2}}}$

$$- \frac{\frac{1}{2}\dot{y} \times \overline{a-y}}{y^{\frac{3}{2}} \times \overline{a-y}|^{\frac{1}{2}}} = \frac{-\frac{1}{2}y\dot{y} - \frac{1}{2}a\dot{y} + \frac{1}{2}y\dot{y}}{y^{\frac{3}{2}} \times \overline{a-y}|^{\frac{1}{2}}} = \frac{-\frac{1}{2}a\dot{y}}{y^{\frac{3}{2}} \times \sqrt{a-y}} =$$

$$\frac{-\frac{1}{2}a}{y^{\frac{3}{2}} \times \sqrt{a-y}} \times \frac{\sqrt{a-y}}{y^{\frac{1}{2}}} = \frac{-a}{2y^2}; \ \therefore \frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{a^{\frac{3}{2}}}{y^{\frac{3}{2}}} \times \frac{2y^2}{a} = 2a^{\frac{1}{2}}y^{\frac{1}{2}}$$

$= 2DO$.

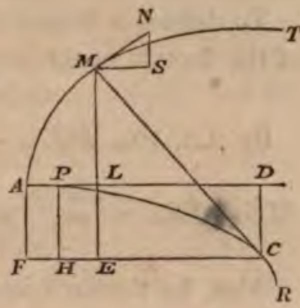
Cor. Let $y=a$; then *RB*, the radius of curvature at the vertex, $= 2DB$.

Ex. 4. To find the chord of curvature in the cycloid at *M*.

Produce *MR* to *V*; take $MV = 4DO$, and draw *VP* perpendicular to *ML* produced. *MP* is the chord required. And by Art. 103. $MP = \dfrac{2\dot{z}^2}{-\ddot{y}} = 2 \times \dfrac{a}{y} \times \dfrac{2y^2}{a} = 4y = 4ML$.

## SECTION II.

### ON THE EVOLUTES OF CURVES.



(105.) Def. A curve *PCR* is said to be the evolute of another curve *AMT*, when it is of such a nature that if a thread *MCR* wrapped round *PCR* were disengaged from it by some power at *M*, whilst the thread remained stretched, the point *M* would trace out the given curve *AMT*.

Cor. 1. *CM* is the radius of curvature at the point *M*, and *C* is the center of curvature.

Cor. 2. If *A* be the vertex of the curve, and *AP* the radius of curvature at the vertex, the evolute *PCR* will leave the axis *APD* at *P*; and since $APC = CM$, the curve line *CP*, which is the length of the evolute, $= CM - AP$.

Cor. 3. The same construction remaining as in Art. 102, if *PH* be drawn perpendicular to *FC*, and *CD* to *AD*, *PH* or *CD* will represent the ordinate of the curve *PCR*, and *PD* or *HC* the abscissa; hence, to determine the nature of the evolute, we must find *PH* and *HC*.

Cor. 4. Since $PH = ME - ML$, and $HC = AD - AP$, the following lines are to be found by the nature of the curve; viz. *ME*, *AP*, and *AD*, or its equal $AL + LD$, or $AL + EC$.

(106.) Hence the Rule.

Having obtained the value of *CM* from one of the expressions for the radius of curvature, make $x$ or $y = 0$; this will determine

*AP* the radius of curvature at the vertex. From *CM* take *AP*; the remainder is *CP* the length of the evolute. Then find *EL* and *CE* from the nature of the curve, and the equation of the evolute will be determined.

### (107.) PROPOSITION.

To deduce a fluxional expression for *LE* and *CE*, in terms of the fluxion of the abscissa, of the ordinate, and of the curve.

By Art. 103, $ME = \frac{\dot{z}^2}{-\dot{x}\ddot{y}} = \frac{\dot{z}^2}{-\ddot{y}}$, if $\dot{x} = 1$; $\therefore$ $LE$, which $=$ $ME - ML$, $= \frac{\dot{z}^2}{-\ddot{y}} - y$.

Also, by similar triangles, *MNS*, *CME*,

$$MN : NS :: CM : CE; \text{ or}$$

$$\dot{z} : \dot{y} :: \frac{\dot{z}^2}{-\dot{x}\ddot{y}} : CE = \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}};$$

or, if $\dot{x} = 1$, $CE = \frac{\dot{y}\dot{z}^2}{-\ddot{y}}$.

### (108.) EXAMPLES.

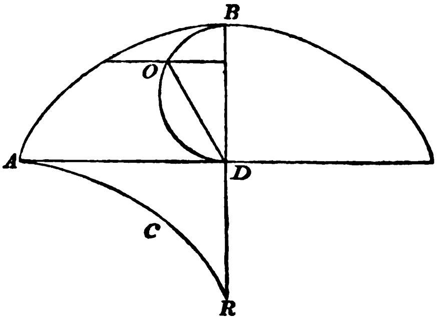
Ex. 1. To find the evolute of the common parabola, and to determine its length.

Let *AMT* represent a parabola. Then (Art. 104. Ex. 1.) $CM = \frac{\overline{4x+a}\,\big|^{\frac{3}{2}}}{2\sqrt{a}}$, and $AP = \frac{a}{2}$; $\therefore$ $CP = \frac{\overline{4x+a}\,\big|^{\frac{3}{2}}}{2\sqrt{a}} - \frac{a}{2} =$ the length of the evolute.

Also $ME = \frac{\dot{z}^2}{-\ddot{y}} = \frac{4x+a}{4x} \times \frac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ (Art. 104. Ex. 1.) $=$ $\frac{4x^{\frac{3}{2}} + ax^{\frac{1}{2}}}{\sqrt{a}}$; $\therefore$ $LE$ or $PH = ME - ML = \frac{4x^{\frac{3}{2}} + ax^{\frac{1}{2}}}{\sqrt{a}} - y =$ $\frac{4x^{\frac{3}{2}} + ax^{\frac{1}{2}}}{\sqrt{a}} - a^{\frac{1}{2}}x^{\frac{1}{2}} = \frac{4x^{\frac{3}{2}}}{\sqrt{a}}$, the ordinate of the evolute.

Again, $AD = AL + LD = x + EC = x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} = x + \frac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}} \times \overline{\frac{4x+a}{4x}} \times \frac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ (Art. 104.) $= x + \frac{4x+a}{2} = 3x + \frac{1}{2}a$; $\therefore HC = AD - AP = 3x$, the abscissa of the evolute. Hence the square of the ordinate $\propto$ the cube of the abscissa, and the evolute is the semi-cubical parabola.

Ex. 2. To find the evolute of the common cycloid, and its length.

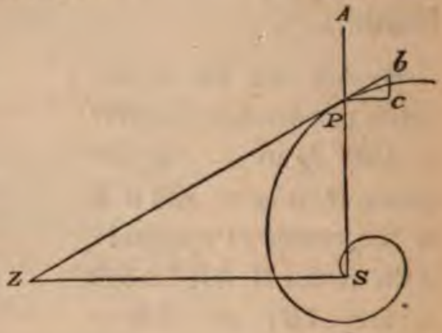


By Art. 104. Ex. 3., the radius of curvature $= 2DO$ in that figure; $\therefore$ at the vertex $A$, it $= 0$; and if $R$ be the center of curvature at the vertex $B$, $RB = 2BD$ = arc $AB$; but $RB = RCA$; $\therefore RCA = AB$; and the evolute $RCA$ is a cycloid similar and equal to $AB$.

# CHAP. XV.

## ON SPIRALS.

### (109.) DEFINITION.



If a line of indefinite length *SA* revolve about *S*, and a point *P* move in it continually from *S*, the point *P* will trace out a spiral; *S* is called the center, and *SP* the distance.

### (110.) PROPOSITION.

To draw a tangent to a spiral at the point *P*.

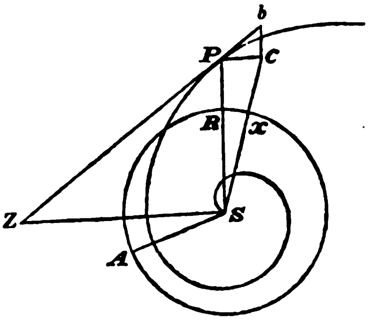
In the revolution of *SP* round *S*, the point *P* has two motions; one in a direction perpendicular to *SP*, and the other in the direction of *SP*. Let *Pc*, and *cb* represent two small spaces, conceived to be described in these two directions, with the velocities at *P* continued uniform; join *Pb*. This case, therefore, coincides with that in Art. 44.; *Pc* represents the fluxion of the abscissa, and *cb* of the ordinate; hence, *Pb* is the fluxion of the curve, and *Pb* produced is a tangent at the point *P*; draw *SZ* perpendicular to *SP*, meeting *bP* produced in *Z*; *SZ* is called the sub-tangent, and its value is to be determined in each case from the nature of the curve.

Let $SP=y$, $bc=\dot{y}$; then, by similar triangles, *SPZ*, *Pbc*,

$$bc : cP :: SP : SZ;$$

$$\text{or, } \dot{y} : cP :: y : SZ; \quad \therefore SZ=\frac{cP\times y}{\dot{y}}.$$

## (111.) PROPOSITION.



The same construction remaining,

If a circle $ARx$ be described with a given radius $SA=a$, and if $AR=x$, and $Rx=\dot{x}$, then $SZ=\frac{y^2\dot{x}}{a\dot{y}}$.

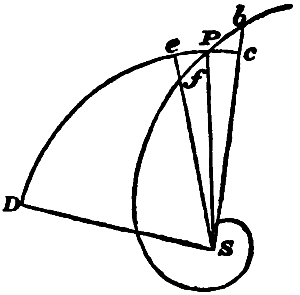
Join $Sc$; then, by similar triangles, $Pbc$, $SPZ$,

$$\begin{array}{l} bc : cP :: SP : SZ \\ cP : Rx :: SP : SR \\ \hline \therefore\ bc : Rx :: SP^2 : SR \times SZ; \end{array}$$

or, $\dot{y} : \dot{x} :: y^2 : a \times SZ$.

$$\therefore\ SZ=\frac{y^2\dot{x}}{a\dot{y}}.$$

(112.) To find the areas of spirals.



Let the spiral $SfP$ be conceived to be described by the uniform angular motion of $SP$; and suppose $Sf$ and $Sb$ to be two distances near $SP$, which make equal angles with it. With $S$ as a center, and $SP$ as radius, describe the circular arc $DPc$. Then $SPc$ being equal to $SPe$, is greater than the preceding increment $SPf$ of the area, and less than the succeeding; also, since $SP$ revolves uniformly, $SPc$ is uniformly described; $\therefore$ $SPc$ is the fluxion of the area $SfP$.

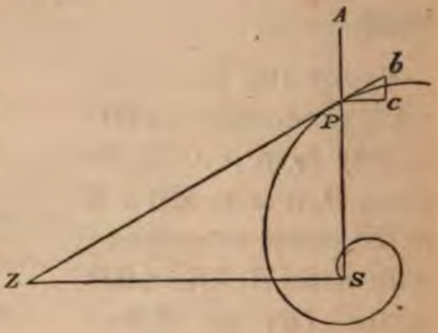
Now if the arc $DP=x$, and $SP=y$, the sector $SPc=\frac{y\dot{x}}{2}$;

$\therefore$ the fluxion of $SfP=\frac{y\dot{x}}{2}$.

# CHAP. XV.

## ON SPIRALS.

### (109.) DEFINITION.



IF a line of indefinite length *SA* revolve about *S*, and a point *P* move in it continually from *S*, the point *P* will trace out a spiral; *S* is called the center, and *SP* the distance.

### (110.) PROPOSITION.

To draw a tangent to a spiral at the point *P*.

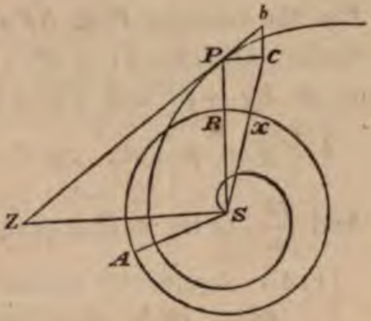
In the revolution of *SP* round *S*, the point *P* has two motions; one in a direction perpendicular to *SP*, and the other in the direction of *SP*. Let *Pc*, and *cb* represent two small spaces, conceived to be described in these two directions, with the velocities at *P* continued uniform; join *Pb*. This case, therefore, coincides with that in Art. 44.; *Pc* represents the fluxion of the abscissa, and *cb* of the ordinate; hence, *Pb* is the fluxion of the curve, and *Pb* produced is a tangent at the point *P*; draw *SZ* perpendicular to *SP*, meeting *bP* produced in *Z*; *SZ* is called the sub-tangent, and its value is to be determined in each case from the nature of the curve.

Let $SP = y$, $bc = \dot{y}$; then, by similar triangles, *SPZ*, *Pbc*,

$$bc : cP :: SP : SZ;$$

$$\text{or, } \dot{y} : cP :: y : SZ; \therefore SZ = \frac{cP \times y}{\dot{y}}.$$

(111.) PROPOSITION.



The same construction remaining,

If a circle $ARx$ be described with a given radius $SA=a$, and if $AR=x$, and $Rx=\dot{x}$, then $SZ = \frac{y^2\dot{x}}{a\dot{y}}$.

Join $Sc$; then, by similar triangles, $Pbc$, $SPZ$,

$$bc : cP :: SP : SZ$$

$$cP : Rx :: SP : SR$$

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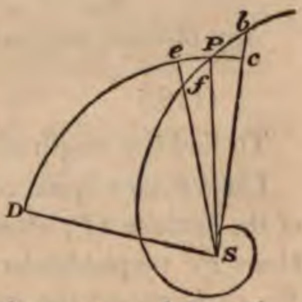
$$\therefore\ bc : Rx :: SP^2 : SR \times SZ;$$

or, $\dot{y} : \dot{x} :: y^2 : a \times SZ.$

$$\therefore\ SZ = \frac{y^2\dot{x}}{a\dot{y}}.$$

(112.) To find the areas of spirals.



Let the spiral $SfP$ be conceived to be described by the uniform angular motion of $SP$; and suppose $Sf$ and $Sb$ to be two distances near $SP$, which make equal angles with it. With $S$ as a center, and $SP$ as radius, describe the circular arc $DPc$. Then $SPc$ being equal to $SPe$, is greater than the preceding increment $SPf$ of the area, and less than the succeeding; also, since $SP$ revolves uniformly, $SPc$ is uniformly described; $\therefore$ $SPc$ is the fluxion of the area $SfP$. Now if the arc $DP=x$, and $SP=y$, the sector $SPc=\frac{y\dot{x}}{2}$;

$\therefore$ the fluxion of $SfP=\frac{y\dot{x}}{2}$.

## CHAP. XV.

### ON SPIRALS.

(109.) **DEFINITION.**

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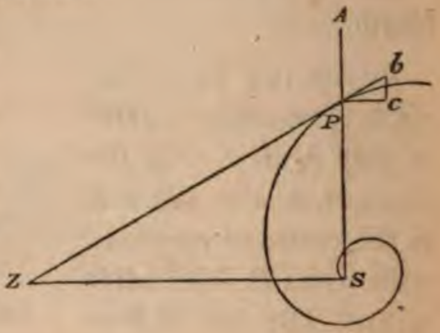


(110.) **PROPOSITION.**

To draw a tangent to a spiral at the point $P$.

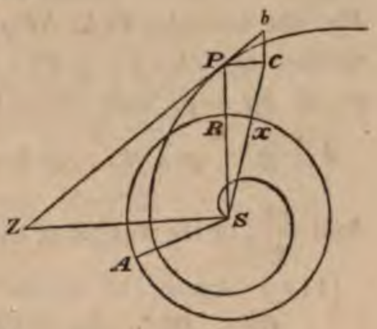
In the revolution of $SP$ round $S$, the point $P$ has two motions; one in a direction perpendicular to $SP$, and the other in the direction of $SP$. Let $Pc$, and $cb$ represent two small spaces, conceived to be described in these two directions, with the velocities at $P$ continued uniform; join $Pb$. This case, therefore, coincides with that in Art. 44.; $Pc$ represents the fluxion of the abscissa, and $cb$ of the ordinate; hence, $Pb$ is the fluxion of the curve, and $Pb$ produced is a tangent at the point $P$; draw $SZ$ perpendicular to $SP$, meeting $bP$ produced in $Z$; $SZ$ is called the sub-tangent, and its value is to be determined in each case from the nature of the curve.

Let $SP=y$, $bc=\dot{y}$; then, by similar triangles, $SPZ$, $Pbc$,

$$bc : cP :: SP : SZ;$$

$$\text{or, } \dot{y} : cP :: y : SZ; \therefore SZ=\frac{cP\times y}{\dot{y}}.$$

### (111.) PROPOSITION.



The same construction remaining,

If a circle $ARx$ be described with a given radius $SA=a$, and if $AR=x$, and $Rx=\dot{x}$, then $SZ = \frac{y^2\dot{x}}{a\dot{y}}$.

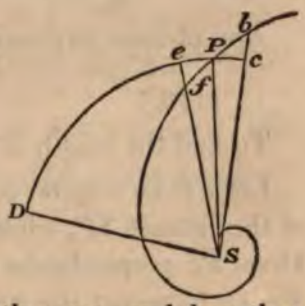
Join $Sc$; then, by similar triangles, $Pbc$, $SPZ$,

$$\begin{array}{l} bc : cP :: SP : SZ \\ cP : Rx :: SP : SR \\ \hline \therefore\ bc : Rx :: SP^2 : SR\times SZ; \end{array}$$

or, $\dot{y} : \dot{x} :: y^2 : a\times SZ$.

$$\therefore\ SZ = \frac{y^2\dot{x}}{a\dot{y}}.$$

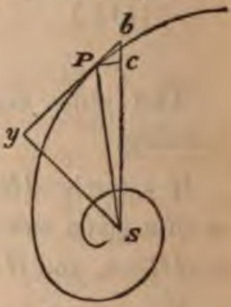
(112.) To find the areas of spirals.



Let the spiral $SfP$ be conceived to be described by the uniform angular motion of $SP$; and suppose $Sf$ and $Sb$ to be two distances near $SP$, which make equal angles with it. With $S$ as a center, and $SP$ as radius, describe the circular arc $DPc$. Then $SPc$ being equal to $SPe$, is greater than the preceding increment $SPf$ of the area, and less than the succeeding; also, since $SP$ revolves uniformly, $SPc$ is uniformly described; $\therefore$ $SPc$ is the fluxion of the area $SfP$. Now if the arc $DP=x$, and $SP=y$, the sector $SPc=\frac{y\dot{x}}{2}$;

$\therefore$ the fluxion of $SfP=\frac{y\dot{x}}{2}$.

(113.) If $bPy$ be a tangent at $P$, and $PSb$ a very small angle, and $Sy$ be drawn perpendicular to $Py$, the triangles $Pcb$, $SPy$ are ultimately similar, and $cb=\dot{y}$; $\therefore Pc : cb :: Sy : Py$; or, if $Sy = p$, and $Py = t$,



$$\dot{x} : \dot{y} :: p : t; \ \therefore \ \dot{x} = \frac{p\dot{y}}{t}.$$

And $\frac{y\dot{x}}{2}$, or the fluxion of the area, $=\frac{py\dot{y}}{2t}$.

(114.) Again, by similar triangles, $Pcb$, $SPy$,

$$Pc : Pb :: Sy : SP;$$

$$\text{or,} \ \dot{x} : \dot{z} :: p : y; \ \therefore \ \frac{y\dot{x}}{2} = \frac{p\dot{z}}{2};$$

and the fluxion of the area $= \frac{p\dot{z}}{2}$.

(115.) If with $S$ as a ~~circle~~ centre, and $SA=a$ as radius, a circle be described cutting $SP$, $SC$, in $R$ and $x$; and $AR = w$; the fluxion of the area $=\frac{y^2\dot{w}}{2a}$. (See the first FIGURE in the preceding page.)

$$\text{For} \ Pc : Rx :: SP : SR;$$

$$\text{or,} \ \dot{x} : \dot{w} :: y : a;$$

$$\therefore \dot{x} = \frac{y\dot{w}}{a}; \ \text{and} \ \frac{y\dot{x}}{2}, \ \text{or the fluxion of the area} = \frac{y^2\dot{w}}{2a}.$$

Any of these expressions may be adopted to find the area.

### (116.) PROPOSITION.

To find the length of spirals. (See the FIGURE in Art. 113.)

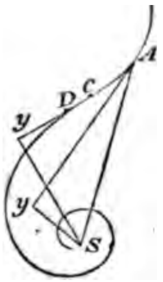
Let $SP$ be a spiral curve described by the uniform revolution of the distance $SP$, whilst the point $P$ moves continually from $S$. Draw $Pc$ perpendicular to $SP$, and $cb$ parallel to it; and let $Pc$, $cb$ represent the uniform velocities of $SP$ and the point $P$; join $Pb$, then $Pb$ is in the direction of a tangent at $P$, and is the fluxion of the curve. Draw $Sy$ perpendicular to the tangent; then, by similar triangles, $Pbc$, $SPy$,

$$Pb : bc :: SP : Py$$

$$\text{or,} \ \dot{z} : \dot{y} :: y : t, \ \ \therefore \ \dot{z} = \frac{y\dot{y}}{t}.$$

### (117.) PROPOSITION.

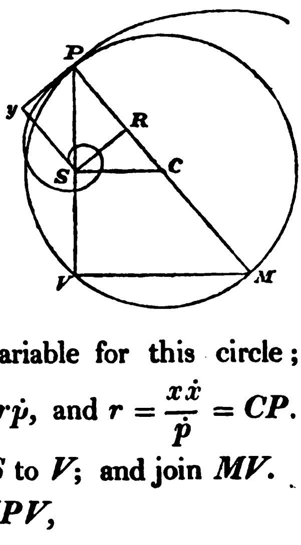
To find the point of contrary flexure.



Let $C$ be the point of contrary flexure; suppose the curve convex from $A$ to $C$, and concave afterward toward $S$. Draw $Sy$ perpendicular to the tangent $Ay$. Then, while $A$ approaches $C$, $Sy$ is increasing; after that time it decreases to $S$. Hence, if the fluxion of the perpendicular be assumed $= 0$, it will determine the point of contrary flexure.

### (118.) PROPOSITION.

To find an expression for the radius, and chord of curvature, in spiral curves.



Let $PVM$ be the circle, and $PCM$ the diameter of curvature; $S$ the center of the spiral, and $C$ that of the circle. Draw $Sy$ perpendicular to the tangent $Py$, $SR$ perpendicular to $PM$, and join $SC$. Let $SP = x$, $Sy = RP = p$, $CP = r$; then, by Euclid,

$$SC^2 = CP^2 + PS^2 - 2CP \times PR$$
$$= r^2 + x^2 - 2rp;$$

Now $SC$ may be considered as invariable for this circle; $\therefore$ its fluxion $= 0$; hence, $0 = 2x\dot{x} - 2r\dot{p}$, and $r = \dfrac{x\dot{x}}{\dot{p}} = CP$.

Next, to find the chord, produce $PS$ to $V$; and join $MV$.

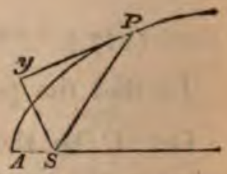
Then, by similar triangles, $SPy$, $MPV$,

$$SP : Sy :: MP : PV;$$

or, $x : p :: \dfrac{2x\dot{x}}{\dot{p}} : PV = \dfrac{2p\dot{x}}{\dot{p}} =$ the chord of curvature.

### (119.) EXAMPLES.

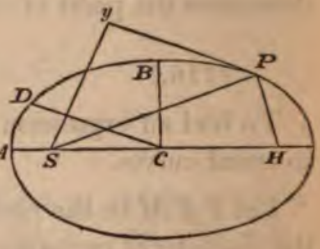
Ex. 1. To find the radius, and chord of curvature of the parabola, considered as a spiral.



Here $p^2 = ax$; $\therefore 2p\dot{p} = a\dot{x}$, and $\frac{x\dot{x}}{\dot{p}} = \frac{2p}{a} \times x = \frac{2p^3}{a^2}$ = the radius of curvature.

Also $\frac{\dot{x}}{\dot{p}} = \frac{2p}{a}$; $\therefore \frac{2p\dot{x}}{\dot{p}} = \frac{4p^2}{a} = \frac{4ax}{a} = 4x$; or the chord $= 4SP$.

Ex. 2. Let the ellipse be considered as a spiral; to find the chord of curvature passing through the focus.



Let $AC = a$, $SP = x$, $Sy = p$;
$CB = b$, $HP = v$;

then, $p^2 = \frac{b^2 \times x}{v}$, by Conic Sections; $\therefore 2p\dot{p} = \frac{b^2 v\dot{x} - b^2 x\dot{v}}{v^2}$; but $x + v = 2a$; $\therefore \dot{x} = -\dot{v}$; hence, $2p\dot{p} = \frac{b^2 \times \overline{v+x} \,.\, \dot{x}}{v^2} = \frac{2ab^2\dot{x}}{v^2}$; $\therefore \frac{\dot{x}}{\dot{p}} = \frac{2pv^2}{2ab^2}$, and $\frac{2p\dot{x}}{\dot{p}} = \frac{4p^2v^2}{2ab^2} = \frac{4 \times b^2xv}{2ab^2} = \frac{2SP \times HP}{AC} = \frac{2CD^2}{AC}$, where $CD$ is the conjugate diameter to $CP$.

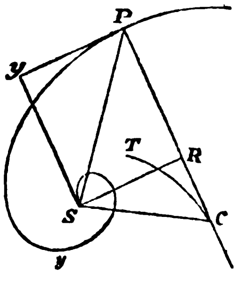
Ex. 3. To find the chord of curvature of the spiral, whose equation is $p = \frac{ax}{\sqrt{a^2+x^2}}$.

Here $\dot{p} = a\dot{x} \times \overline{a^2 + x^2}|^{-\frac{1}{2}} - x\dot{x} \times ax \times \overline{a^2 + x^2}|^{-\frac{3}{2}} = \frac{a^3\dot{x} + ax^2\dot{x} - ax^2\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}} = \frac{a^3\dot{x}}{\overline{a^2 + x^2}|^{\frac{3}{2}}}$; $\therefore$ the chord of curvature, or $\frac{2p\dot{x}}{\dot{p}} = \frac{2x \times \overline{a^2 + x^2}}{a^2}$.

(120.) PROPOSITION.

To determine the evolutes of spiral curves.

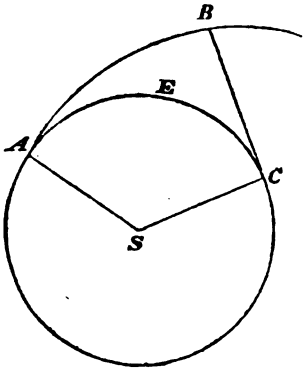
Let $SVP$ represent any spiral curve; $CP$ the radius of curvature at the point $P$; $S$ the pole of the spiral; $Sy$ perpendicular to a tangent at $P$, and $SR$ perpendicular to $CP$. Join $SC$, and let $CT$ be the evolute; then, by Art. 118, $CP = \frac{x\dot{x}}{\dot{p}}$; and $CP$ corrected, if necessary, gives the length of the evolute. Next, find $CR = CP - PR$ the tangent of the evolute; then, $CS$ its ordinate, which $= \sqrt{CR^2 + RS^2}$. From these two, the nature of the curve will be known.

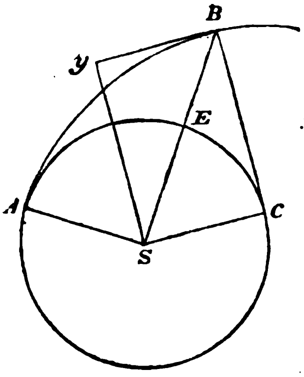


## ON THE INVOLUTE OF A CIRCLE.

### (121.) Definition.

The involute $AB$ of a circle is described by the extremity $B$ of a string unwinding itself from the circumference of a circle $AEC$.





Cor. 1. A small arc at $B$ may be considered as a circular arc, whose radius is $CB$; and $CB$ is therefore perpendicular to the curve or to the tangent $By$.

Cor. 2. If $SY$ be drawn perpendicular to $By$, and $SC$ be joined, $BCSy$ is a parallelogram; and $By = SC$, the radius of the circle.

### (122.) PROBLEMS.

PROB. 1. To find the area of the involute.

Let $SB = y$, $By = SC = r$, $Sy = p$; then, $Sy = \sqrt{y^2 - r^2}$; $\therefore$ the fluxion of the area, or $\frac{py\dot{y}}{2t}$ (Art. 113.) $= \frac{\sqrt{y^2 - r^2} \times y\dot{y}}{2r}$; and the fluent $= \frac{\overline{y^2 - r^2}|^{\frac{3}{2}}}{6r} +$ cor.

Let $y = SA = r$; then the area $= 0$; therefore cor. $= 0$; $\therefore$ the corrected fluent or the area $SAB = \frac{\overline{y^2 - r^2}|^{\frac{3}{2}}}{6r} = \frac{Sy^3}{6SC}$.

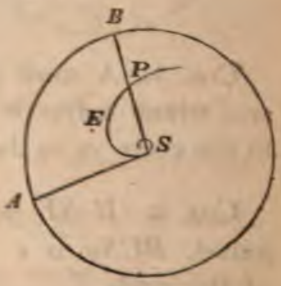
COR. Since $CB = CEA$, $\frac{CB \times SC}{2} = \frac{CEA \times SC}{2}$, or the area of the triangle $SCB =$ the sector $ASC$; hence the area $ASCB -$ the triangle $SCB =$ the area $ASCB -$ the sector $ASC$; that is, the area $ASB =$ the area $AECB$; consequently the area $AECB = \frac{Sy^3}{6SC}$.

PROB. 2. To find the length of the involute.

Here $\dot{z} = \frac{y\dot{y}}{t} = \frac{y\dot{y}}{r}$; $\therefore z = \frac{y^2}{2r} +$ cor.; but when $y = r$, $z = 0$; $\therefore$ cor. fluent $= \frac{y^2 - r^2}{2r} = \frac{Sy^2}{2SC}$.

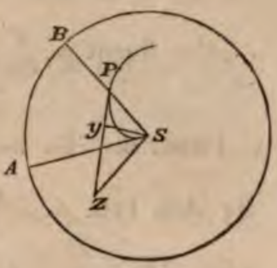
## ON THE SPIRAL OF ARCHIMEDES.

(123.) DEF. If with $S$ as a pole, a spiral $SEP$ be described of such a nature, that the distance $SP$ always bears a given ratio to the arc $AB$ of a given circle, whose center is $S$, $SEP$ is called the Spiral of Archimedes.



(124.) **Problems.**

**Prob. 1.** To draw a tangent to this spiral.



Let $PZ$ represent the tangent at $P$, $SZ$ the sub-tangent; $SY$ a perpendicular on $PZ$. Then if $AB = x$, $SP = y$, $SA = a$, $SZ = \frac{y^2\dot{x}}{a\dot{y}}$ (Art. 111.); but here $x : y :: c : d$, a given ratio; $\therefore\ \dot{x} = \frac{c\dot{y}}{d}$; $\therefore\ SZ = \frac{y^2}{a\dot{y}} \times \frac{c\dot{y}}{d} = \frac{cy^2}{ad} = \frac{y^2}{b}$, if $\frac{ad}{c}$ be assumed $= b$.

**Prob. 2.** To find the length of the tangent $PZ$.

$PZ^2 = SP^2 + SZ^2 = y^2 + \frac{y^4}{b^2} = \frac{y^2 \times \overline{b^2+y^2}}{b^2}$; therefore $PZ = \frac{y \times \sqrt{b^2+y^2}}{b}$.

**Prob. 3.** To find $PY$.

We have $ZP : PS :: PS : PY$;

$$\text{or, } \frac{y}{b} \times \sqrt{b^2+y^2} : y :: y : PY = \frac{by}{\sqrt{b^2+y^2}}.$$

**Prob. 4.** To find the perpendicular $SY$.

By similar triangles, $SPY$, $SPZ$.

$$PZ : SZ :: SP : SY;$$

$$\text{or, } \frac{y}{b} \times \sqrt{b^2+y^2} : \frac{y^2}{b} :: y : SY = \frac{y^2}{\sqrt{b^2+y^2}}.$$

**Prob. 5.** To find its area.

One expression for the fluxion of the area is $\frac{y^2\dot{w}}{2a}$ (Art. 115.) where $\dot{w}$ is the fluxion of a circular arc of radius $a$, or in this case the fluxion of $AB$; $\therefore$ here the expression is $\frac{y^2\dot{x}}{2a}$.

Now, $x : y :: c : d$; $\therefore\ \dot{x} = \frac{c\dot{y}}{d}$;

$\therefore$ the fluxion of the area $= \frac{y^2}{2a} \times \frac{c\dot{y}}{d} = \frac{cy^2\dot{y}}{2ad} = \frac{y^2\dot{y}}{2b}$;

$\therefore$ the fluent $= \frac{y^3}{6b}$; cor. $= 0$; and the area $= \frac{y^3}{6b}$.

PROB. 6. To find its length.

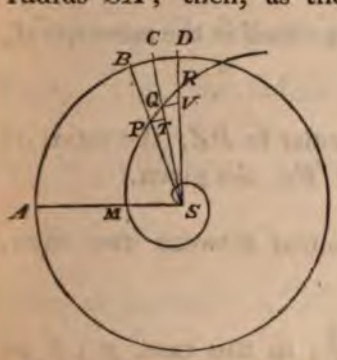
By Art. 116. $\dot{x} = \frac{y\dot{y}}{t}$; in this case, $t = \frac{by}{\sqrt{b^2 + y^2}}$; $\therefore\ \dot{x} = \frac{y\dot{y} \times \sqrt{b^2 + y^2}}{by} = \frac{\dot{y} \times \sqrt{b^2 + y^2}}{b}$; whose fluent (*fl.* 58.) $= \frac{1}{2b} \times \sqrt{y^4 + b^2y^2} + \frac{1}{2}\, b \times \text{hyp. log.} \frac{y + \sqrt{y^2 + b^2}}{b}$.

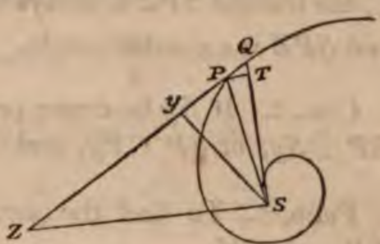
PROB. 7. To find the radius of curvature, or the length of the evolute.

By Art. 118, the radius of curvature $= \frac{x\dot{x}}{\dot{p}}$, where $x = SP$; that is, in this case it $= \frac{y\dot{y}}{\dot{p}}$. Now $p = \frac{y^2}{\sqrt{y^2 + b^2}}$; $\therefore\ \dot{p} = \frac{y^3\dot{y} + 2b^2y\dot{y}}{\overline{y^2 + b^2}|^{\frac{3}{2}}}$ (Art. 20. Ex. 23.); hence, the radius of curvature, or $\frac{y\dot{y}}{\dot{p}} = \frac{\overline{y^2 + b^2}|^{\frac{3}{2}}}{y^2 + 2b^2}$ + cor. Let $y = 0$, this expression $= \frac{b}{2}$; $\therefore$ cor. $= -\frac{b}{2}$; that is, the radius of curvature, or the length of the evolute reckoned from the vertex, $= \frac{\overline{y^2 + b^2}|^{\frac{3}{2}}}{y^2 + 2b^2} - \frac{b}{2}$.

## ON THE LOGARITHMIC SPIRAL.

(125.) Def. Let $ABD$ be a circle described with a given radius $SA$; then, as the arc $AB$ of that circle increases in arithmetic progression, if the distance $SP$ increase in geometric, the point $P$ will trace out the logarithmic spiral.





Cor. 1. Let $SP$, $SQ$, $SR$ be assumed indefinitely near each other, and equidistant, so that the arc $BC = CD$. Draw $PT$, $QV$, perpendicular to $SQ$, $SR$. Then, since $SP : SQ :: SQ : SR$, dividendo $SP : QT :: SQ : RV$, or $QT \propto SP$. That is, if $SP = y$; since $QT$ ultimately becomes $\dot{y}$, $\dot{y} \propto y$.

Cor. 2. The distance $SP$ being supposed to revolve uniformly round $S$, the angle $PSQ$ is constant, or $\frac{PT}{SP}$ is a constant quantity; hence, $PT \propto SP \propto QT$.

### (126.) Problems.

Prob. 1. To draw a tangent to the logarithmic spiral at any point $P$.

The subtangent $SZ = \frac{PT \times y}{\dot{y}}$ (Art. 110.) Now $PT : QT$

($\dot{y}$) in a given ratio, :: $a : b$; $\therefore$ $SZ = \frac{ay}{b}$. Hence, draw $SZ$ perpendicular to $SP$, and equal to $\frac{ay}{b}$; join $ZP$; it is the tangent required.

Cor. 1. $SP : SZ :: y : \frac{ay}{b} :: b : a$, a given ratio; $\therefore$ the triangle $SPZ$ is always similar to itself in the same spiral, and $SPZ$ is a constant angle.

Cor. 2. If $Sy$ be drawn perpendicular to $PZ$, the ratios of $SP : Sy$, of $SP : Py$, and of $Sy : Py$, are given.

Prob. 2. To find the area contained between two rays, $SP$ and $SM$.

The fluxion of a spiral area $= \frac{py\dot{y}}{2t}$; in this case, $p : t$ in a given ratio, :: $a : b$; $\therefore$ the fluxion of the area $= \frac{ay\dot{y}}{2b}$; and the fluent $= \frac{ay^2}{4b}$ + cor.; let $SP = SM = d$; then the area $= \frac{a}{4b} \times \overline{y^2 - d^2}$; or the area $SPM = \frac{a}{4b} \times \overline{SP^2 - SM^2}$.

Cor. 1. The area contained between $SP$ and $SM \propto SP^2 - SM^2$.

Cor. 2. If $SM = 0$, or the whole spiral area be found between $SP$ and the pole $S$, it $= \frac{a}{4b} \times SP^2 = \frac{Sy}{4Py} \times SP^2 =$ (since $Sy : Py :: SZ : SP$) $\frac{SZ}{4SP} \times SP^2 = \frac{SZ \times SP}{4}$ = half the area of the triangle $SPZ$.

Prob. 3. To find its length.

The fluxion of the length $= \frac{y\dot{y}}{t}$. In this case, $y : t$ in a constant ratio, :: $m : n$; $\therefore$ $\frac{y}{t} = \frac{m}{n}$, and $\dot{z} = \frac{m\dot{y}}{n}$; $\therefore$ $z = \frac{my}{n}$ +

cor. Let $SP = SM = d$; then $PM = \frac{m}{n} \times \overline{SP - SM} = \frac{SP}{Py} \times \overline{SP - SM}$.

Cor. 1. The arc $PM \propto SP - SM$.

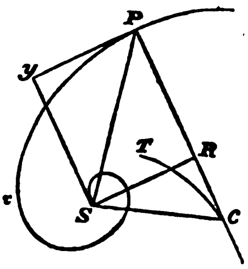
Cor. 2. The whole arc to the pole $S = \frac{SP^2}{Py}$ = (since $ZP : SP :: SP : Py$) the tangent $ZP$.

Prob. 4. To find the radius and chord of curvature.

By Art. 118., the radius of curvature $= \frac{x\dot{x}}{\dot{p}}$, where $SP = x$; that is, in this case, where $SP = y$, it $= \frac{y\dot{y}}{\dot{p}}$. But $y : p$, in a given ratio, $:: m : r$; $\therefore \frac{\dot{y}}{\dot{p}} = \frac{m}{r}$, and $\frac{y\dot{y}}{\dot{p}} = \frac{my}{r}$; or the radius of curvature $= \frac{m \times SP}{r}$.

Again, by Art. 118., the chord of curvature $= \frac{2p\dot{x}}{\dot{p}}$; in this case $= \frac{2p\dot{y}}{\dot{p}}$. But since $y : p :: m : r$; $\therefore y : p :: \dot{y} : \dot{p}$, and $\frac{\dot{y}}{\dot{p}} = \frac{y}{p}$; hence, the chord of curvature $= 2p \times \frac{y}{p} = 2y = 2SP$.

Prob. 5. To determine the evolute $CT$.



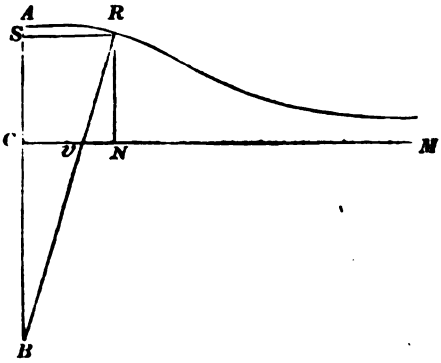
By the nature of the evolute, $CP$ the radius of curvature $= CT$; and $CP = \frac{m \times SP}{r}$ (by the last Prob.), or $CP : SP :: m : r$. But $SP : PR$ or $Sy :: m : r$; $\therefore CP : SP :: SP : PR$; hence (Euclid, B. vi.) the triangles $CPS$, $SPR$ are similar; $\therefore PSC$ is a right angle, and the angle $SCP$ = the angle $SPy$; and $SC : CP$ in the given ratio of $Py$ to $SP$. Therefore the evolute $CT$ is a logarithmic spiral similar to $SvP$.

Cor. A line $SC$ drawn from $S$ perpendicular to $SP$ passes through $C$, the center of curvature.

# CHAP. XVI.

## ON THE CONCHOID OF NICOMEDES.

(127.) DEF. LET $CM$ be a line of indefinite length, given in position; if about some point $B$ taken without it, an indefinite line $BR$ revolve, cutting $CM$ in $v$, and $vR$ be always taken of the same length, the point $R$ will trace out the conchoid.



(128.) **PROBLEMS.**

**PROB. 1.** To find the equation.

Draw $BCA$ and $RN$ perpendicular to $CM$, and $RS$ perpendicular to $AB$. Take $BC=a$, $CA=vR=b$, $CN=x$, $NR=y$; then, by similar triangles, $BSR$, $vRN$,

$$BS : SR :: RN : Nv, \text{ or } a+y : x :: y : Nv = \frac{xy}{a+y};$$

and $vR^2 = vN^2 + NR^2$, or $b^2 = y^2 + \frac{x^2y^2}{a+y^2}$; $\therefore \overline{a+y}|^2 \times b^2 = \overline{a+y}|^2 \times y^2 + x^2y^2$, or $\overline{a+y}|^2 \times \overline{b^2-y^2} = x^2y^2$, the equation.

**PROB. 2.** To draw a tangent to the conchoid.

Since by the equation $xy = \overline{a+y} \times \sqrt{b^2-y^2}$; $\therefore x\dot{y} + y\dot{x} = \dot{y} \times \sqrt{b^2-y^2} - \frac{y\dot{y} \times \overline{a+y}}{\sqrt{b^2-y^2}}$; $\therefore y\dot{x} = \dot{y} \times \sqrt{b^2-y^2} - \frac{y\dot{y} \times \overline{a+y}}{\sqrt{b^2-y^2}}$

$-x\dot{y} = \left(\text{since } x = \frac{\overline{a+y} \times \sqrt{b^2-y^2}}{y}\right) \dot{y} \times \sqrt{b^2-y^2} - \frac{y\dot{y} \times \overline{a+y}}{\sqrt{b^2-y^2}}$ $- \frac{\overline{a+y} \times \sqrt{b^2-y^2} \times \dot{y}}{y} =$ by reduction $- \frac{\overline{y^3+ab^2}}{y \times \sqrt{b^2-y^2}} \times \dot{y}$; hence the subtangent, or $\frac{y\dot{x}}{\dot{y}} = - \frac{y^3+ab^2}{y \times \sqrt{b^2-y^2}}$, a negative quantity; therefore the tangent and the vertex are on contrary sides of the ordinate, and the value of the subtangent, thus taken, is $\frac{ab^2+y^3}{y \times \sqrt{b^2-y^2}}$.

Prob. 3. To find the area.

By the preceding problem, $y\dot{x} = \frac{-y^3\dot{y}}{y \times \sqrt{b^2-y^2}} - \frac{ab^2\dot{y}}{y\sqrt{b^2-y^2}} = \frac{-y^2\dot{y}}{\sqrt{b^2-y^2}} - \frac{ab^2\dot{y}}{y \times \sqrt{b^2-y^2}}$; and the area or $f.\ y\dot{x} =$ corr. $- \frac{1}{2}b$ × a circular arc of radius $b$, and sine $y + \frac{y \times \sqrt{b^2-y^2}}{2}$ (Fluent 15.) $- \frac{ab}{2} \times$ hyp. log. $\frac{b-\sqrt{b^2-y^2}}{b+\sqrt{b^2-y^2}}$ (Art. 43). Now the area $= 0$, when $y = b$; therefore the area $ARNC =$ ($\frac{1}{2}b$ × a circular arc of radius $b$, and sine $b$, or) $\frac{1}{2}b \times \overline{\text{arc of a quad.} - \text{arc whose rad. is } b \text{ and sine } y + \frac{y \times \sqrt{b^2-y^2}}{2}}$ $- \frac{ab}{2} \times$ hyp. log. $\frac{b-\sqrt{b^2-y^2}}{b+\sqrt{b^2-y^2}}$.

Prob. 4. To find the content of the solid generated by the revolution of the conchoid about its axis $CM$.

Since $y\dot{x} = \frac{-y^3\dot{y}}{y \times \sqrt{b^2-y^2}} - \frac{ab^2\dot{y}}{y \times \sqrt{b^2-y^2}}$; ∴ $py^2\dot{x}$, or the fluxion of the content $= -p \times : \frac{y^3\dot{y}}{\sqrt{b^2-y^2}} + \frac{ab^2\dot{y}}{\sqrt{b^2-y^2}}$; and the content itself $= \text{cor.} + p \times \frac{2 \times \overline{b^2-y^2}|^{\frac{3}{2}}}{3} + py^2 \times \sqrt{b^2-y^2}$ (*f.* 17.) $- pab \times$ a circular arc of radius $b$ and sine $y$ (Art. 44.) let $y = b$, the content $= 0$; therefore the correct fluent is $pab \times \overline{\text{the arc of a quadrant} - \text{arc of radius } b \text{ and sine } y} + \frac{2p}{3} \times \overline{b^2-y^2}|^{\frac{3}{2}} + py^2 \times \sqrt{b^2-y^2}$, the content of the solid generated by the revolution of *ARNC* about *CN*. Let $y=0$; then we get the content of the whole solid when its axis is infinite $= pab \times \text{the arc of a quadrant} + \frac{2pb^3}{3} = pb^2 \times \overline{\frac{1}{2}pa + \frac{2b}{3}}$.

Prob. 5. To find the point of contrary flexure.

Since $y\dot{x} = -\frac{y^3+ab^2}{y \times \sqrt{b^2-y^2}} \times \dot{y}$; $\therefore \dot{x} = -\frac{y^3+ab^2}{y^2 \times \sqrt{b^2-y^2}} \times \dot{y}$.

Assume $\dot{y}$ constant or $=1$; then, $\ddot{x} = \frac{2b^4a - b^2y^3 - 3b^2ay^2}{\overline{b^2y^3 - y^5} \times \sqrt{b^2-y^2}} = 0$; $\therefore 2b^4a - 3b^2ay^2 - b^2y^3 = 0$, or $y^3 + 3ay^2 - 2ab^2 = 0$, an equation from which the value of $y$, and therefore of $x$, may be determined.

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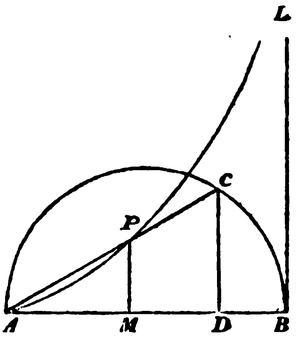
## ON THE CISSOID OF DIOCLES.

(129.) Let *AB* be the diameter of a semi-circle *ACB*, *D* and *M* two points in it equally distant from *B* and *A*; from *D* and *M* draw *DC*, *MP*, perpendicular to *AB*; and join *AC*, cutting *MP* in *P*. The point *P* traces [illegible]soid.



COR. *BL* drawn perpendicular to *AB* is an asymptote to the curve *AP*.

## (130.) PROBLEMS.

PROB. 1. To find the equation to the cissoid of Diocles.

By the properties of the circle, $AD \times DB = DC^2$; but $AD \times DB = AM \times MB$; $\therefore$ $AM \times MB = DC^2$. Also $AM : MP :: (AD)\ MB : DC$; $\therefore$ $DC = \frac{MP \times MB}{AM}$; $\therefore$ $DC^2 = \frac{MP^2 \times MB^2}{AM^2}$; consequently $\frac{MP^2 \times MB^2}{AM^2} = AM \times MB$; $\therefore$ $MP^2 \times MB = AM^3$; that is, if $AB = a$, $AM = x$, $MP = y$, $y^2 \times \overline{a-x} = x^3$.

PROB. 2. To draw a tangent to the cissoid.

The equation is $y^2 = \frac{x^3}{a-x}$; $\therefore 2y\dot{y} = \frac{3x^2\dot{x} \times \overline{a-x} + x^3\dot{x}}{\overline{a-x}|^2} = \frac{\overline{3ax^2 - 3x^3 + x^3} \times \dot{x}}{\overline{a-x}|^2}$; $\therefore$ $\frac{\dot{x}}{\dot{y}} = \frac{2y \times \overline{a-x}|^2}{3ax^2 - 2x^3}$; $\therefore$ $\frac{y\dot{x}}{\dot{y}}$ the subtangent $= \frac{2y^2 \times \overline{a-x}|^2}{3ax^2 - 2x^3} = \frac{2x^3 \times \overline{a-x}}{3ax^2 - 2x^3} = \frac{2x \times \overline{a-x}}{3a - 2x}$.

PROB. 3. To find the area.

Since $y = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}$; $\therefore$ $y\dot{x} = \frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a-x}} = \frac{x^2\dot{x}}{\sqrt{ax-x^2}}$, whose fluent (*f*. 23.) $= \frac{3a}{4} \times$ a circular arc of radius $= \frac{1}{2}a$, and versed sine $x - \underbrace{x \times \sqrt{ax-x^2}}_{2} - \frac{3a}{4} \times \sqrt{ax - x^2}$; which vanishes when $x = 0$; and when $x = a$, it $= \frac{3a}{4} \times$ the arc of a semicircle; $\therefore$ for both sides of the axis it $= \frac{3a}{2} \times$ the arc of a semi-circle = three generating circles.

**Prob. 4.** To find the content of the solid generated by the rotation of the cissoid about its axis *AB*.

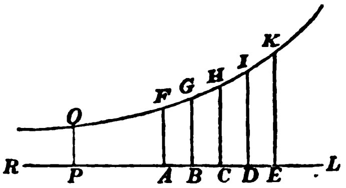
The $f. py^2\dot{x} = f. \frac{px^3\dot{x}}{a-x} = -\frac{1}{3}px^3 - \frac{1}{2}pax^2 - pa^2x + pa^3 \times$ hyp. log. $\frac{a}{a-x}$. (Fluent 3.)

◆

## ON THE LOGARITHMIC CURVE.

### (131.) Definition.

If on the indefinite line *AL*, the parts *AB, BC, CD, DE*, &c. be taken equal to each other, and ordinates *AF, BG, CH*, &c. be drawn perpendicular to *AL*, and in geometrical progression, the curve *FHK*, which passes through all their extremities, is called the Logarithmic Curve.



**Cor.** The axis *LA* produced, is an asymptote to the curve; for since the terms of a decreasing geometric series never become accurately equal to nothing, a distant ordinate, as *OP*, never is accurately equal to nothing, though it decreases *sine limite.*

### (132.) Problems.

**Prob. 1.** To find the equation to this curve.

By the nature of logarithms, any abscissa *AD* is the logarithm of the ordinate *DI*, in a system which depends upon the magnitude of *AF*, and *BG*; the part *AB* being given. Let $AB=1$, the logarithm of *BG*; then if $BG=a$, $AD=x$, $DI=y$,

$1 : x ::$ log. of $a$ : log. of $y$; but $1 =$ log. of $a$; $\therefore x =$ log. of $y$; that is, $x \times 1$, or $x \times$ log. of $a$, $=$ log. of $y$; $\therefore y = a^x$ the equation.

**Prob. 2.** To draw a tangent to the logarithmic curve.

Let $y=a^x$; $\therefore$ log. $y=x\times$ log. $a$; and $\frac{\dot{y}}{y}=\dot{x}\times\text{log.}\,a$; $\therefore\frac{y\dot{x}}{\dot{y}}=$ $=\frac{1}{\text{log.}\,a}=$ a constant quantity.

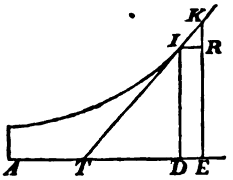
**Cor. 1.** If $\frac{1}{\text{log.}\,a}=m$, $\frac{y\dot{x}}{\dot{y}}=m$; and $y\dot{x}=m\dot{y}$, which may be taken as another equation to the curve.

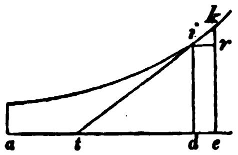
**Cor. 2.** In any system, where $x$ is the logarithm of $y$ to a modulus $m$, $\dot{x}=m\times\frac{\dot{y}}{y}$ (Art. 41.); hence, in the logarithmic curve, the sub-tangent is the modulus of the system.

### (133.) Proposition.

If two ordinates in one logarithmic curve be in the same ratio with two ordinates in another, the abscissas are as the sub-tangents.

Let $AE$, $ae$ be the abscissas, $ET$ and $et$ the sub-tangents,





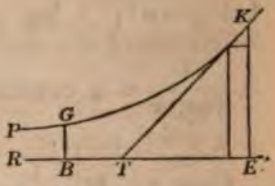
$KE$, $ID$ in one curve, and $ke$, $id$ in the other, ordinates indefinitely near to each other, and in a given ratio;

Then $KE : ID :: ke : id$, by supposition;

$\therefore KE : KR :: ke : kr$, or $y : \dot{y} :: Y : \dot{Y}$;

$\therefore\frac{\dot{y}}{y}=\frac{\dot{Y}}{Y}$; but $\dot{x}=m\times\frac{\dot{y}}{y}$; $\therefore\dot{x}$ in one case : $\dot{X}$ in the other $::m\times\frac{\dot{y}}{y}:M\times\frac{\dot{Y}}{Y}::m:M$, and $x:X::m:M$, or $AE:ae::TE:te$.

**Prob. 3.** To find the area included between any two given ordinates, as $BG$, $EK$.



Since $y\dot{x}=m\dot{y}$; $\therefore f.\ y\dot{x}=my+$ corr. Let the area $=0$, when $y=BG=a$; then the area included between $BG$ and $EK=my-ma=m\times\overline{EK-BG}$.

**Cor. 1.** If $BG$ go off *ad infinitum*, and become evanescent, the whole area included between the asymptote $ER$ and the curve $=m\times EK=$ sub-tangent $\times KE=$ twice the triangle $EKT$.

**Cor. 2.** In the same logarithmic curve, the area included between any two ordinates : the area included between any other two :: the difference of the two first : the difference of the two last.

**Cor. 3.** The whole area included from $KE$ between the asymptote and the curve varies as $KE$.

**Prob. 4.** To find the content of the solid generated by the revolution of the area $BGKE$ about its axis $BE$.

Since $y\dot{x}=m\dot{y}$; $\therefore f.\ py^2\dot{x}=f.\ mpy\dot{y}=mp\frac{y^2}{2}+$ corr. But the content $=0$, if $y=BG=a$; $\therefore$ the content corrected $=\frac{mp}{2}\times\overline{y^2-a^2}$; and the content included between $BG$ and $EK$ $=\frac{mp}{2}\times\overline{KE^2-BG^2}$.

**Cor. 1.** Suppose $GB$ to go off *ad infinitum*; then the content of the whole solid generated by the revolution of the infinitely extended area $KPRE$ about its axis, $=\frac{mp}{2}\times KE^2$ $=\frac{m}{2}\times$ the area of a circle, whose radius is $KE=\frac{1}{2}$ the content of a cylinder whose altitude is the sub-tangent, and radius of the base equal to the ordinate of the curve.

Cor. 2. The solid generated by the revolution of $BGKE =$ $\frac{1}{2}$ the cylinder, whose altitude is the sub-tangent, and base the difference of the circles, whose radii are $EK$ and $BG$.

Cor. 3. The content of the infinitely extended solid varies as $KE^2$, where the sub-tangent is the same.

Prob. 5. To find the length of the logarithmic curve included between any two ordinates $GB$, $KE$.

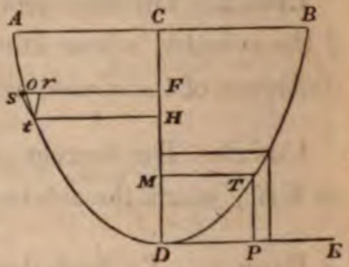
Here $\dot{z} : \dot{y} :: KT : KE :: \sqrt{m^2+y^2} : y$; therefore $\dot{z} = \frac{\dot{y} \times \sqrt{m^2+y^2}}{y}$; $\therefore$ by *f.* 60, $z$ or $KG = \sqrt{m^2+y^2} + \frac{m}{2} \times$ hyp. log. $\frac{\sqrt{m^2+y^2}-m}{\sqrt{m^2+y^2}+m} - \sqrt{m^2+a^2} - \frac{m}{2} \times$ hyp. log. $\frac{\sqrt{m^2+a^2}-m}{\sqrt{m^2+a^2}+m}$, corrected for the value of $y = a$.

Prob. 6. To find the surface of the solid generated by the revolution of any part $KG$ of the logarithmic curve about its axis.

By the last Example, $\dot{z} = \frac{\dot{y} \times \sqrt{m^2+y^2}}{y}$; $\therefore$ $2py\dot{z}$, the fluxion of the surface, $= 2p \times \dot{y} \times \sqrt{m^2+y^2}$, of which the fluent, (by *fl.* 58.) when properly corrected, $= p \times \sqrt{y^4+m^2y^2}$ $- p \times \sqrt{a^4+m^2a^2} + pm^2 \times$ hyp. log. $\frac{y+\sqrt{m^2+y^2}}{a+\sqrt{m^2+a^2}} =$ the surface required; the values of $a$ and $m$ being the same as before.

## ON THE CATENARY.

(134.) To find the curve, into which a flexible chain of uniform density and thickness would form itself, if suffered to hang freely from two points, $A$ and $B$.



Let $ADB$ be the curve, $D$ the lowest point; draw the axis $DC$ perpendicular to the horizon; let $Ht$, $Fo$, be two ordinates, $ts$ a tangent at $t$, and $tr$ parallel to $CD$. Now the chain having assumed this form, it is immaterial whether the part $Dt$ be considered as flexible or rigid. On the last supposition, it is kept at rest by three forces; by $BD$ in the direction $De$ or $sF$, by $At$ in the direction $ts$, and by its gravity in the direction $rt$; therefore, by Mechanics, these forces are as the sides of the triangle $str$. Let the tension of $BD = a$, $Dt = z$, $DH = x$, $Ht = y$; then $a : z :: \dot{y} : \dot{x}$; $\therefore\ a\dot{x} = z\dot{y}$, and $\dot{y}^2 = \frac{a^2\dot{x}^2}{z^2}$; $\therefore\ \dot{z}^2$, or $\dot{x}^2 + \dot{y}^2$, $= \dot{x}^2 + \frac{a^2\dot{x}^2}{z^2}$; hence, $z\dot{z} = \sqrt{z^2+a^2} \times \dot{x}$; $\therefore\ \dot{x} = \frac{z\dot{z}}{\sqrt{z^2+a^2}}$; and $x = \sqrt{z^2+a^2}$ + cor. Let $x = 0$, $z = 0$; $\therefore$ cor. $= -a$; $\therefore\ x = \sqrt{z^2+a^2} - a$; $\therefore\ a + x = \sqrt{z^2+a^2}$, and $x^2 + 2ax = z^2$, the equation to the curve.

Cor. Since $\dot{y} = \frac{a\dot{x}}{z} = \frac{a\dot{x}}{\sqrt{2ax+x^2}}$; $y = a \times$ hyperb. log. $(x + a \overset{+}{\cancel{\times}} \sqrt{2ax+x^2}.)$

## Problems.

Prob. 1. To draw a tangent to the catenary.

By the nature of the curve, $a\dot{x} = z\dot{y}$; $\therefore\ \frac{\dot{x}}{\dot{y}} = \frac{z}{a}$; $\therefore\ \frac{y\dot{x}}{\dot{y}} = \frac{y \times z}{a} = \frac{y \times \sqrt{2ax+x^2}}{a}$ = the sub-tangent.

**Prob.** 2. To find its area. (See the preceding **Figure.**)

By the equation, $z\dot{y}=a\dot{x}$;

$$\therefore\ z^2\dot{y}^2=a^2\dot{x}^2=a^2\times\overline{\dot{z}^2-\dot{y}^2}=a^2\dot{z}^2-a^2\dot{y}^2;$$

$$\therefore\ \overline{z^2+a^2}\dot{y}^2=a^2\dot{z}^2,$$

$$\text{or } \overline{a+x}|^2\times\dot{y}^2=a^2\dot{z}^2;$$

$$\therefore\ \overline{a+x}.\dot{y}=a\dot{z},$$

$$\text{or } a\dot{y}+x\dot{y}=a\dot{z};$$

$\therefore x\dot{y}$, the fluxion of the external area $DTB$, $=a\dot{z}-a\dot{y}$; and the area $BDT=az-ay$, and cor. $=0$; $\therefore DMT=xy-az+ay$ $=\overline{a+x}\times y-a\sqrt{2ax+x^2}$.

**Prob.** 3. To find the content of the solid generated by the revolution of the catenary about its axis.

The fluent of $py^2\dot{x}=py^2x-f.\,2pxy\dot{y}$; and the fluent of $2py\times x\dot{y}=$ the fluent of $2pay\dot{z}-$ the fluent of $2pay\dot{y}$; since, by the last Article, $x\dot{y}=a\dot{z}-a\dot{y}$;

$$\therefore\ f.\,2pxy\dot{y}=f.\,2pay\dot{z}-pay^2;$$

$$\text{and } f.\,2pay\dot{z}=\quad 2payz-f.\,2paz\dot{y};$$

$$=\quad 2payz-2pa^2x, \text{ by writing for } z\dot{y}$$

its equal $a\dot{x}$; $\therefore$ the content $=py^2\dot{x}+pay^2-2payz+2pa^2x$, which needs no correction.

**Prob.** 4. To find the surface of the solid generated by the revolution of the catenary.

The fluent of $2py\dot{z}=2pyz-f.\,2pz\dot{y}$. But $z\dot{y}=a\dot{x}$; $\therefore$ the fluent $=2pyz-2pax$, which needs no correction.

**Prob.** 5. To find the radius of curvature of the catenary.

By Art. 100., the radius of curvature $=\dfrac{\dot{z}^3}{-\ddot{y}}$, when $\dot{x}$ is constant, and $= 1$. Now $z=\sqrt{2ax+x^2}$; ∴ $\dot{z}=\frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}}$ $=\frac{a+x}{\sqrt{2ax+x^2}}$; ∴ $\dot{z}^3=\frac{\overline{a+x}|^3}{\overline{2ax+x^2}|^{\frac{3}{2}}}$. Also $\dot{y}=\frac{a\dot{x}}{\sqrt{2ax+x^2}}=\frac{a}{\sqrt{2ax+x^2}}$; ∴ $\ddot{y}=a\times\frac{\overline{a+x}}{\overline{2ax+x^2}|^{\frac{3}{2}}}$; ∴ the radius of curvature $=\frac{\overline{a+x}|^3}{a\times\overline{a+x}}=\frac{\overline{a+x}|^2}{a}$.

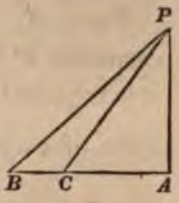
**Cor.** At the vertex, where $x=0$, the radius of curvature $=a$.

# CHAP. XVII.

## ON THE ATTRACTIONS OF BODIES.

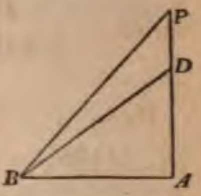
### (135.) PROBLEMS.

PROB. 1. To determine the attraction of a corpuscle *P* towards a right line *AB*, in a direction perpendicular to that line; the attracting force of each particle being supposed to vary inversely as the square of the distance.



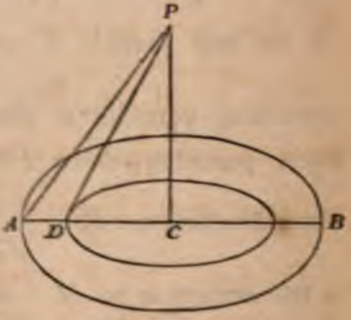
Let *PA* perpendicular to $AB=a$; *AC*, a variable part of *AB*, $=x$; then $PC=\sqrt{a^2+x^2}$. Now, since the force of attraction varies as $\frac{1}{\overline{\text{dist.}}|^2}$, the force which draws *P* toward *C* may be represented by $\frac{1}{PC^2}$. But the force of attraction toward *C* : that toward *A* by resolution of force :: *PC* : *PA* :: $\sqrt{a^2+x^2} : a$; therefore the force of attraction toward *A* for one particle $C = \frac{a}{\overline{a^2+x^2}|^{\frac{3}{2}}}$. And if the number of attracting particles at the distance $x=\dot{x}$, the attraction from these particles will be $\dot{x}$ times as great; hence the fluxion of the attraction to $AC=\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$; and its fluent, or $\frac{x}{a\times\sqrt{a^2+x^2}}$, is the attraction to *AC*; and the attraction of *P* to the whole line $AB=\frac{AB}{PA\times PB}$.

**Prob. 2.** To find the attraction of a line $PA$ to $AB$; $PA$ being perpendicular to $AB$, and the force of each particle varying as $\frac{1}{\text{dist.}^2}$.



Let $AB=a$, $AD$ a variable part of $AP=x$; join $BD$. Then, by the last Case, the attraction of a corpuscle $D$ to $AB=\frac{a}{x\times\sqrt{a^2+x^2}}$. Therefore the fluxion of this attraction $=\frac{a\dot{x}}{x\times\sqrt{a^2+x^2}}$; and the fluent, or the attraction itself, $=\frac{1}{2}$ hyp. log. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$ + corr. (Art. 43.) $=\frac{1}{2}$ hyp. log. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$ $-\frac{1}{2}$ hyp. log. $\frac{0}{2a}$, when $x=PA=\frac{1}{2}$ hyp. log. $\frac{BP-BA}{BP+BA}-$ $\frac{1}{2}$ hyp. log. $\frac{0}{2AB}$, an infinite quantity.

**Prob. 3.** Let $C$ be the center of a circle $ABE$, and a corpuscle $P$ be situate in the line $PC$ perpendicular to its plane; it is required to determine the attraction of $P$ to the circle, the force of each particle varying as $\frac{1}{D^2}$.



Let $PC=a$, $PD=x$; then $CD=\sqrt{x^2-a^2}$, and the force of attraction of $P$ to $D$ being represented by $\frac{1}{PD^2}$, that in the direction $PC=\frac{PC}{PD^3}$ (Art. 135. Prob. 1.) Now if $p=3.14159$ &c. the area of a circle whose radius is 1, the area of the circle of radius $CD=p\times\overline{x^2-a^2}$; and its fluxion $=2px\dot{x}$; hence the fluxion of the attraction of $P$ toward the circle $=\frac{a}{x^3}\times 2px\dot{x}=$

$2pax^{-1}\dot{x}$; whose fluent $= 2pa \times -\frac{1}{x} + \text{cor.} = 2pa \times \overline{\frac{1}{a} - \frac{1}{x}}$ $= 2p \times \overline{1 - \frac{a}{x}} =$ for the whole circle $2p \times \overline{1 - \frac{PC}{PA}}$.

Cor. The force of attraction of $P$ to the whole circle varies as $1 - \frac{PC}{PA}$.

Prob. 4. To determine the same, when the force of attraction varies as $\overline{D}|^n$.

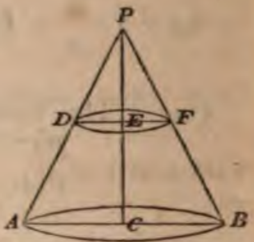
Here, as before, $2px\dot{x}$ is the fluxion of the variable circle, and $\frac{a}{x} \times x^n$ or $ax^{n-1}$ represents the force of a particle $D$, in the direction $PC$; hence, the fluxion of the attraction to the circle $= 2pax^n\dot{x}$; and the fluent $= 2pa \times \frac{x^{n+1}}{n+1} + \text{cor.} =$ $2pa \times \frac{x^{n+1} - a^{n+1}}{n+1}$, which varies as $\frac{ax^{n+1} - a^{n+2}}{n+1}$, or varies for the whole circle as $\frac{PC \times PA^{n+1} - PC^{m+2}}{n+1}$.

Cor. 1. If the diameter of the circle be increased in infinitum, and $n$ be a negative number greater than 1, the attraction varies as $-\frac{PC}{PA^{n-1}} + \frac{1}{PC^{n-2}}$, or as $\frac{1}{PC^{n-2}}$, because the first term vanishes.

Cor. 2. If the radius of the circle be infinite, and the force vary as $\frac{1}{D^3}$, the attraction varies as $\frac{1}{PC^0}$; or is constant, whatever be the length of $PC$.

Prob. 5. Let $P$ represent a corpuscle placed in the vertex of a cone $PAB$; to determine the attraction to the cone by a force varying as $\frac{1}{D^2}$.

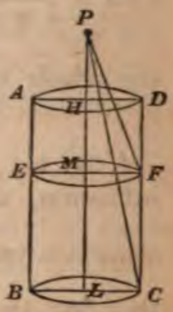
The attraction to the circle *DEF* varies as $1 - \frac{PE}{PF}$; and since $\frac{PE}{PF}$ in the cone is a constant quantity, the attraction to every section parallel to the base is the same; hence, the whole attraction to the cone varies as $\overline{1 - \frac{PE}{PF}} \times$ the number of sections, or as $\overline{1 - \frac{PC}{PB}} \times PC$.



Cor. For similar cones, the attraction is as the height.

Prob. 6. The law of the force remaining the same, to find the attraction of a cylinder *ABCD* upon a corpuscle *P* situated in its axis produced.

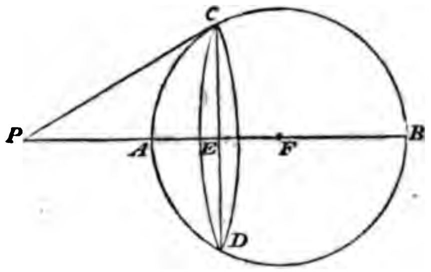
Let $PM = x$, $MF = a$, and $PF = \sqrt{a^2+x^2}$. The attraction to the circle *EMF* is as $1 - \frac{PM}{PF}$, or as $1 - \frac{x}{\sqrt{a^2+\dot{x}^2}}$; $\therefore$ the fluxion of the attraction is proportional to $\dot{x} - \frac{x\dot{x}}{\sqrt{a^2+x^2}}$; and the $fl. = x - \sqrt{a^2+x^2} + \text{cor.} = PM - PF + \text{cor.}$ : $= PM - PF - PH + PD$, or for the whole cylinder $= PL - PH - PC + PD = HL - PC + PD$.



Cor. If the diameter be infinite, $PC = PD$; and the attraction is proportional to *HL*.

Prob. 7. To find the attraction of a corpuscle *P* to a sphere, the attractive force of each particle varying as $\frac{1}{D^2}$.

Draw *PFB* passing through *F* the center of the sphere, and



let $CED$ be a section of the sphere perpendicular to $AB$. Take $AF = a$, $PF = b$, $PA = c = b - a$, $PE = y$, $PC = c + x$; then, $AE = y - c$, and $EB = 2a - y + c$. Now $AE \times EB = EC^2 = PC^2 - PE^2$; that is, $\overline{y-c} \times \overline{2a - y + c} = \overline{c+x}|^2 - y^2$; from which equation $y = \frac{2ac + 2c^2 + 2cx + x^2}{2a + 2c}$ (since $b = a + c$) $= \frac{2bc + 2cx + x^2}{2b}$.

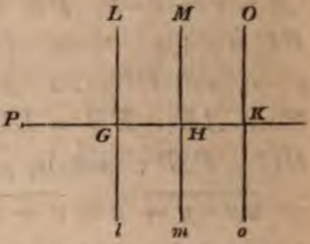
Hence the attraction to the circle is as $1 - \frac{2bc + 2cx + x^2}{2b \times \overline{c+x}}$, or as $\frac{2ax - x^2}{b \times \overline{c + x}}$; and the fluxion of the attraction to the sphere is proportional to $\frac{2ax - x^2}{b \times \overline{c+x}} \times \dot{y}$, that is, to $\frac{2ax - x^2}{b \times \overline{c+x}} \times \frac{\overline{c+x}.\dot{x}}{b}$, or to $\frac{2ax\dot{x} - x^2\dot{x}}{b^2}$; whose fluent $= \frac{ax^2 - \frac{1}{3}x^3}{b^2}$, which needs no correction. Let $x = 2a$; then the attraction to the whole sphere is as $\frac{4a^3}{3b^2}$, or varies as $\frac{a^3}{b^2}$.

Cor. 1. If the corpuscle be at the surface of the sphere, or if $b$ be any multiple of $a$, the attraction varies as $a$.

Cor. 2. Since the contents of spheres vary as $a^3$, the attraction varies as the content divided by the square of the distance from the center; and is the same as if all the matter of the sphere were collected into the center.

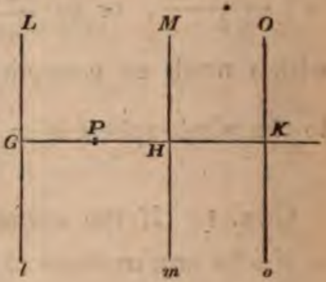
Prob. 8. To find the attraction of a particle $P$ towards a plane $LloO$ infinitely extended, the force of each particle varying inversely as the $\overline{\text{dist.}}|^n$.

### Case 1.



Let *MHM* represent a section of the infinite plane perpendicular to *PHK*; the attraction to the plane *MHM*, considered as circular, is as $\frac{1}{PH^{n-2}}$, or if $PH = x$, as $\frac{1}{x^{n-2}}$; $\therefore$ the fluxion of the attraction is as $\frac{\dot{x}}{x^{n-2}}$; and the fluent $=$ cor. $-$ $\frac{1}{\overline{n-3}\,.\,x^{n-3}} = \frac{1}{\overline{n-3}\,.\,PG^{n-3}} - \frac{1}{\overline{n-3}\,.\,PH^{n-3}}$; when *PH* becomes infinite, it $= \frac{1}{\overline{n-3}\,.\,PG}$; or the whole attraction varies as $\frac{1}{PG^{n-3}}$.

### Case 2.



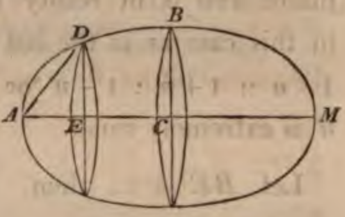
If *P* be placed within the plane, and *PH* be taken $=$ *PG*, the particle *P* will be equally attracted each way towards the planes bounded by *LGl*, *MHm*, or they will destroy the effect of each other; hence, *P* may be considered as attracted only by the infinite plane *MHmoKO*; and the attraction to this plane by the last Case, is as $\frac{1}{PH^{n-3}}$, or as $\frac{1}{PG^{n-3}}$.

Cor. 1. If $n = 3$, the fluent fails; for the fluxion of the attraction is as $\frac{\dot{x}}{x}$.

Cor. 2. If the plane be finite, but *PG* extremely small, if compared with *PH*, the force of attraction is still proportional to $\frac{1}{PG^{n-3}}$.

**Def.** An oblong spheroid is generated by the revolution of an ellipse round its major axis.

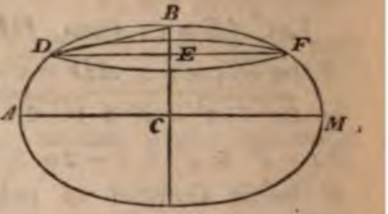
**Prob. 9.** A particle of matter is placed in the pole $A$ of an oblong spheroid, whose major axis : minor :: $1 : 1-n$, where $n$ is very small; it is required to compare the attraction of this particle to the center of the spheroid, with its attraction to the center of the sphere described round the axis major, the force of each particle varying as $\frac{1}{D^2}$.



Let $AE=x$; then, $DE=\overline{1-n}\times\sqrt{2x-x^2}$ by the property of the ellipse; $\therefore AD^2=x^2+\overline{1-n}^2\times\overline{2x-x^2}=x^2+\overline{1-2n+n^2}\times\overline{2x-x^2}=x^2+\overline{1-2n}\times\overline{2x-x^2}$, $n^2$ being omitted as indefinitely small, $=x^2+2x-x^2-4nx+2nx^2=2x-4nx+2nx^2$; $\therefore AD=\sqrt{2x-4nx+2nx^2}$. Hence the attractive force to the circle, whose diameter is $2DE$, $=1-\frac{x}{\sqrt{2x-4nx+2nx^2}}$; and the fluxion of the attraction of the spheroid is proportional to $\dot{x}-\frac{x\dot{x}}{\sqrt{2x-4nx+2nx^2}}$; therefore the attraction itself is as the fluent of this quantity, or proportional to $x-\frac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3}-\frac{2^{\frac{1}{2}}nx^{\frac{3}{2}}}{3}+\frac{nx^{\frac{5}{2}}}{5\times 2^{\frac{1}{2}}}$ (*fl.* 91.) which needs no correction. Let $x=2$; then the attraction to the whole spheroid is proportional to $2-\frac{4}{3}-\frac{4n}{3}+\frac{4n}{5}$, which varies as $\frac{2}{3}-\frac{8n}{15}$, or as $1-\frac{4n}{5}$. If $n=0$, the spheroid becomes a sphere; hence, the attraction of this spheroid on a particle at $A$; the attraction of the sphere described round $AM$ on the same particle $:: 1-\frac{4n}{5} : 1$.

DEF. An oblate or compressed spheroid is generated by the revolution of an ellipse round the minor axis.

PROB. 10. A particle is placed in the pole $B$ of an oblate spheroid, whose minor axis : major :: 1 : $1+n$, where $n$ is very small; it is required to compare the attraction of this particle toward the center of the spheroid, with its attraction toward the center of the sphere described round its minor axis, the force of each particle varying as $\frac{1}{D^2}$.



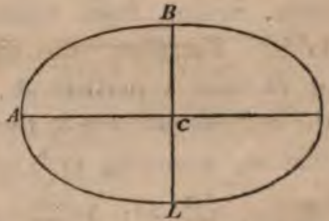
The ratio of the major to the minor axis is in reality the same in this case as in the last; for 1 : $1-n$ :: $1+n$ : $1-n^2$ or 1, since $n^2$ is extremely small.

Let $BE = x$; then, $DE^2 = \overline{1+n}|^2 \times \overline{2x - x^2}$ by the property of the ellipse; $\therefore BD^2 = x^2 + \overline{1+n}|^2 \times \overline{2x - x^2} = x^2 + \overline{1 + 2n + n^2} \times \overline{2x - x^2} = x^2 + \overline{1 + 2n} \times \overline{2x - x^2} = x^2 + 2x + 4nx - 2nx^2 - x^2 = 2x + 4nx - 2nx^2$; $\therefore BD = \sqrt{2x + 4nx - 2nx^2}$; and the attraction to the circle $DEF$ is as $1 - \frac{x}{\sqrt{2x+4nx-2nx^2}}$; $\therefore$ the fluxion of the attraction to the spheroid varies as $\dot{x} - \frac{x\dot{x}}{\sqrt{2x+4nx-2nx^2}}$, and the fluent $=$ (*fl.* 92.) $x - \frac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3} - \frac{nx^{\frac{5}{2}}}{5 \times 2^{\frac{1}{2}}} + \frac{2^{\frac{1}{2}}nx^{\frac{3}{2}}}{3}$, which needs no correction. Let $x = 2$; then the attraction to the whole spheroid is as $2 - \frac{4}{3} - \frac{4n}{5} + \frac{4n}{3}$, or as $\frac{2}{3} + \frac{8n}{15}$, which varies as $1 + \frac{4n}{5}$. If $n = 0$, the spheroid becomes a sphere round the axis minor; hence, the attraction of the oblate spheroid on a particle at $B$ : attraction of the sphere described round the axis minor on the same particle :: $1 + \frac{4n}{5}$ : 1.

COR. 1. Since these spheroids, by supposition, approach nearly to spheres, they may, without sensible error, be assumed for spheres containing the same quantity of matter; and their attractions, *cæteris paribus*, will be proportional to their quantities of matter. But (Art. 51. Ex. 4. COR. 2.) the oblong spheroid : the oblate :: the oblate : the circumscribed sphere; hence, the attraction of the oblong spheroid on a particle at $A$ : the attraction of the oblate on the same particle :: the attraction of the oblate : the attraction of the circumscribed sphere upon it. Therefore, the attraction of the oblate spheroid on a particle at $A$ : the attraction of the circumscribed sphere on the same particle :: $\sqrt{\text{the attraction of the oblong spheroid}}$ : $\sqrt{\text{the attraction of the circumscribed sphere}}$ :: $\sqrt{1-\frac{4n}{5}}$ : $1$ :: $1-\frac{2n}{5}$ : $1$.

COR. 2. By help of the preceding Propositions, since the Earth is an oblate spheroid, we can determine the ratio of its polar and equatorial diameters.

Let $ABL$ represent the earth; $AC$ and $BC$ the equatorial and polar semi-diameters. Then by the preceding Cases, the attraction of the compressed spheroid on a particle at $B$ : the attraction of the inscribed sphere upon the same particle :: $1+\frac{4n}{5}$ : $1$.



The attraction of the inscribed sphere on a particle at $B$ : the attraction of the circumscribed sphere on the same at the surface, or at $A$, :: $1$ : $1+n$. (Chap. xvii. Prob. 7. Cor. 1.)

The attraction of the circumscribed sphere on a particle at $A$ : the attraction of the compressed spheroid on an equal particle at $A$ :: $1$ : $1-\frac{2n}{5}$.

Therefore the attraction of the compressed spheroid on a particle at $B$ : the attraction of the same spheroid on an equal particle at $A$ :: $1+\frac{4n}{5} : 1+\frac{3n}{5}$.

Now these attractions must be proportional to the weights of equal quantities of matter at those points; and if the particles are not equal, their weights are as the weights of equal particles, and their magnitudes jointly. Hence, if the axes are in the ratio of $1-n : 1$, and particles be taken proportional to the axes in each, we have, the weight of $1-n$ particles on the spheroid at $B$ : the weight of a particle at $A$ :: $\overline{1+\frac{4n}{5}} \times \overline{1-n} : 1+\frac{3n}{5}$

$$:: 1-\frac{n}{5} : 1+\frac{3n}{5},$$

the powers of $n$ being omitted. And dividendo,

The weight of a particle at $A$ – the weight of $1-n$ particles at $B$ : the weight of a particle at $A$ :: $\frac{4n}{5} : 1+\frac{3n}{5}$:

Now if the figure of the Earth be caused by rotation round its axis $BL$, since $\overline{1-n}$ times a particle at $B$ must be in equilibrio with a particle at $A$, the excess of weight of one above the other must be counterbalanced by the centrifugal force, arising from rotation, and is consequently equivalent to it. Therefore, the difference of weight of $1-n$ particles at $B$, and a particle at $A$ : the weight of a particle at $A$ :: the centrifugal force at the equator : the force of gravity there; that is, according to NEWTON (Lib. III.) :: 1 : 289.

Hence, $\frac{4n}{5} : 1+\frac{3n}{5} :: 1 : 289$,

or $\frac{4n}{5} \times 289 = \frac{5+3n}{5}$. From this equation, $n = \frac{1}{230}$ nearly. Consequently, the polar diameter [redacted] is to the equatorial :: $1-n : 1$, is as $1-\frac{1}{230} : 1$,

or :: 229 : 230.

# CHAP. XVIII.

## ON LOGARITHMS.

(136.) ALL numbers in arithmetic may be expressed by the powers of 10. Thus, if $R=10$, $R^2=100$, &c.; the number $5364=5R^3+3R^2+6R+4$. A mixed number, as $45.678=4R+5R^0+6R^{-1}+7R^{-2}+8R^{-3}$. Vulgar fractions, when transformed into decimals, may be expressed in the same way. Thus, $\frac{2}{3}$ or its equal .666 &c. $=6R^{-1}+6R^{-2}+6R^{-3}+$&c.

We may also express all numbers, as near as we please, by a single power of any positive number whatever, except unity. Let the numbers 2 and 10 be taken for examples; then 1, 2, 3, &c. may be expressed by the powers of these numbers. Thus,

| | |
|---|---|
| $1=2^0$ | $1=10^0$ |
| $2=2^1$ | $2=10^{.30103}$ |
| $3=2^{1.58496}$ &c. | $3=10^{.47712}$ |
| $4=2^2$ | &c. |
| &c. | $10=10^1$. |

Hence, if $r$ be assumed some determinate number, $n$ an indefinite positive number, some other number $N$ may always be found such that $r^N=n$. In every case of this kind, $N$ is called the Logarithm of $n$; the logarithms, which are derived by giving a determinate value to $r$, constitute a system of logarithms, and $r$ is the base of that system.

COR. If $e$ represent a number, whose logarithm is 1, and $x$ be the logarithm of some other number as $b$, $e^x=b$.

(137.) In every system of logarithms, the logarithm of 1 must = 0. For if $r^N=n$, and $n$ be assumed = 1, $N$ must = 0; for $r^0=1$.

In every system the logarithm of the base is unity. For let $r=n$; then $N$ must be unity.

The logarithms of any two given numbers have the same ratio in every system. For let $r^N=n$, and $r^B=b$; then $r^{NB}=n^B$, and $r^{NB}=b^N$; $\therefore n^B=b^N$, and $n=b^{\frac{N}{B}}$. Now $r$ is not found in this equation; therefore the value of $\frac{N}{B}$ depends only on the numbers $n$ and $b$.

In every system the base of which is greater than unity, the logarithm of a whole or mixed number is positive.

If not, let $n=r^{-N}$, that is $=\frac{1}{r^N}$; then since $\frac{1}{r^N}$ is a proper fraction, $n$ must be a fraction; which is impossible, for $n$ is by hypothesis greater than unity; therefore $n=r^N$.

In every system where the base is greater than unity, the logarithm of a proper fraction is negative.

If not, let $n=r^N$, where $N$ is positive; since $r^N$ is greater than 1, $n$ is greater than 1; but it is by hypothesis a proper fraction, which is impossible; therefore $n=r^{-N}$.

(138.) Logarithms are also considered as measures of ratios. Thus the ratio of 81 to 3, may be considered as made up of the ratios of 81 to 27, of 27 to 9, and of 9 to 3; which three ratios are equal to each other, and the ratio of 81 to 3 is said to be triple the ratio of 9 to 3. In the same manner, the ratio of 100 to 1 is twice the ratio of 10 to 1; of 1000 to 1, three times that ratio, &c. And if the numbers $A$, $B$, $C$, $D$, are continued proportional, the ratio of $A : B$ being equal to that of $B : C$, &c., then the ratio of $A : D$ is considered as made up of three equal ratios. Hence, ratios may be compared in respect to magnitude; thus, if two ratios can be resolved, one into 5 equal ratios, and the other into 8 of the same ratio, the magnitude of one ratio : the magnitude of the other :: 5 : 8.

In Briggs' system, the measure of the ratio of 10 : 1, or the logarithm of 10 is unity. Napier took the number 2.718282; hence, in the two systems, the logarithms of the ratios are expressed by different numbers. In a given system, the measure of the same ratio is obviously the same; of double the ratio, the measure is double, &c. A ratio of equality has no magnitude; for it has no effect in addition or subtraction; a ratio *majoris inæqualitatis* compounded with another increases it; a ratio *minoris inæqualitatis* diminishes it. If then the measure of the ratio which a greater term bears to a less be positive, the measure of the ratio which a less term bears to a greater is negative; and the measure of a ratio of equality $= 0$.

When this expression, the logarithm of $1+x$, is used, it it denotes the measure of the ratio of $1+x : 1$. The logarithm of $\frac{A}{B}$, is the excess of the logarithm of $A : 1$ above that of $B : 1$, or is the measure of the ratio of $A : B$.

(139.) These being the principles which apply to logarithms in general, the following Propositions are intended to shew the rules by which the logarithm corresponding to any number, or the number corresponding to any logarithm, may be deduced, the one from the other. The quantity $m$ is called the modulus (Art. 41.); it is the measure of some given ratio [redacted], and serves as a standard to which other measures may be referred.

### (140.) Problems.

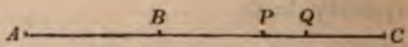
Prob. 1. A number being given, it is required to determine its logarithm.

Let $1+x$ be the number, $y$ its logarithm, and $m$ the modulus; then (Art. 41.) $\dot{y} = \frac{m\dot{x}}{1+x} = m \times \overline{\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} +}$ &c. by actual division; $\therefore\ y = m \times \overline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +}$ &c., which needs no correction; because, if $x=0$, $y=0$; for the number $1+x$ becomes 1, and the logarithm of 1 is 0.

(141.) This proposition is the same with the first in COTES' *Harmonia Mensurarum.* In the following illustration, $AB$ is assumed $= 1$, $BC = x$, and $m$ the modulus; $BP$ is a variable part of $BC$, and $PQ = \dot{x}$.

To find the measure of any proposed ratio.

Let it be required to determine the measure of the ratio between $AC$ and $AB$ (of $1+x : 1$).



Let the difference $BC$ be conceived to be divided into innumerable very small parts as $PQ$ ($\dot{x}$); then the ratio between $AC$ and $AB$ will be divided into as many very small ratios between $AQ$ and $AP$; and if the magnitude of the ratio between $AQ$ and $AP$ be given, by division the ratio of $PQ : AP$ is also given, and therefore the given magnitude of the ratio between $AQ$ and $AP$, may be expressed by the given quantity $\frac{PQ}{AP}$ (that is, $\dot{y} = \frac{\dot{x}}{1+x}$, if $m=1$). If $AP$ remain the same, and $PQ$ be supposed to be increased or diminished in any proportion, the ratio of $AQ$ to $AP$ will be increased or diminished in the same proportion; thus, if $PQ$ be doubled, tripled, &c., or if its value be changed to one half, or three halves of its former value, the ratio will become the duplicate or triplicate, the sub-duplicate or sub-triplicate; it may therefore be still expressed by the quantity $\frac{PQ}{AP}$ (*i. e.* if $1+x$ be invariable, and $\dot{x}$ be doubled, tripled, &c., $\dot{y}$ will be doubled, tripled, &c.); or, if we take some constant quantity $m$, the measure of the ratio between $AQ$ and $AP$ will be expressed by the fraction $\frac{m \times PQ}{AP}$ $\left(\dot{y} = \frac{m\dot{x}}{1+x}\right)$. This measure will have different magnitudes, and will be accommodated to different systems, according to the value of the assumed quantity $m$, which may hence be called the modulus of the system. Now, since the sum of all the ratios between $AQ$ and $AP$ is equal to the proposed ratio of $AC : AB$, so the sum of all the measures $\frac{m \times PQ}{AP}$ (to be found by the known methods) will be equal to the measure required; or, $y = \int . \frac{m\dot{x}}{1+x} = m \times \overline{x - \frac{1}{2}x^2 + \&c.}$

Cor. 1. The measure of any given ratio as $1+x : 1$, where $x$ is given, is as the modulus $(m)$ of the system.

Cor. 2. Since the logarithm of 2 in the common system, is ,3010300; and its logarithm in the hyperbolic is ,6931472, we have ,6931472 : ,3010300 :: 1 (the modulus in the hyperbolic system) : $m$, the modulus in the common system; $\therefore m =$ ,43424948.

Cor. 3. Since the hyperbolic logarithm : the common logarithm :: 1 : $m$ (,43424 &c.), the hyperbolic logarithm $= \frac{\text{the common logarithm}}{m}$; and the common logarithm $= m \times$ the hyperbolic.

Cor. 4. Since $m$ may be assumed of any value, we may, to the same number, have as many different systems of logarithms as we please.

Cor. 5. Since $y$ the measure of the given ratio $1+x : 1$ varies as the modulus $m$, $\frac{y}{m}$ is constant. That ratio whose measure is $m$, is called by Cotes the Modular Ratio.

(142.) By Article 140, $y = m \times \overline{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.}$ Now the smaller $x$ is assumed, the quicker will this series converge. If $x = 1$, $y = m \times \overline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c.} =$ the logarithm of 2 in a system whose modulus is $m$. If $m = 1$, $y = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c. =$ the hyperbolic logarithm of 2.

### (143.) Prob. 2.

To determine the measure of the ratio between $z+x$ and $z-x$, where $z$ is a constant and $x$ a variable quantity; or to find the logarithm of $\frac{z+x}{z-x}$.

Let $y$ be the measure required, $m$ the modulus; then $\dot{y} = 2m \times \frac{z\dot{x}}{z^2 - x^2} = 2m \times \frac{\dot{x}}{z - \frac{x^2}{z}} = 2m \times : \frac{\dot{x}}{z} + \frac{x^2\dot{x}}{z^3} + \frac{x^4\dot{x}}{z^5} + \&c.$;

$\therefore$ the measure $y = 2m \times : \frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} + \&c.$ Cotes, *Schol.* 1.

Cor. 1. If the sum of two quantities is $z$, and their difference $x$, and we assume $2M \times \frac{x}{z} = A$, $A \times \frac{x^2}{z^2} = B$, $\frac{Bx^2}{z^2} = C$, $\frac{Cx^2}{z^2} = D$, &c., the measure of the ratio of the former quantity to the latter $= A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D +$ &c.

Cor. 2. If $z = 1$, $y = 2m \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.}$ = the measure of the ratio of $1+x : 1-x$, or the logarithm of $\frac{1+x}{1-x}$ to a modulus $m$.

Cor. 3. If $m$ also $= 1$, $y = 2 \times \overline{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.}$ = the hyperbolic logarithm of $\frac{1+x}{1-x}$. Let $x = \frac{1}{3}$, then $\frac{1+x}{1-x} = 2$; and $y$ the hyperbolic logarithm of $2 = 2 \times : \frac{1}{3} + \frac{1}{3} \times \frac{1}{3^3} + \frac{1}{5} \times \frac{1}{3^5} + \&c.$, of which the value deduced from the first 7 terms of the series = 0,6931472. This series converges much quicker than that contained in Art. 142.

### (144.) PROB. 3.

Given a logarithm, to find its number.

Let $1+x$ be the number, and $y$ its logarithm; then $\dot{y}=\frac{m\dot{x}}{1+x}$; $\therefore\ \dot{y}+x\dot{y}=m\dot{x}$, and $\dot{y}+x\dot{y}-m\dot{x}=0$. Assume a series $ay+by^2+cy^3+dy^4+$ &c. $=x$; then $\dot{x}=a\dot{y}+2by\dot{y}+3cy^2\dot{y}+$&c.; $\therefore$ by substituting in the equation $\dot{y}+x\dot{y}-m\dot{x}=0$, these values of $x$ and of $\dot{x}$, we have $$\left.\begin{array}{l}\dot{y}+ay\dot{y}+by^2\dot{y}+\text{\&c.}\\ -ma\dot{y}-2mby\dot{y}-3mcy^2\dot{y}-\text{\&c.}\end{array}\right\}=0;$$ hence, $1-ma=0$, or $a=\frac{1}{m}$; $a-2mb=0$, or $b=\frac{a}{2m}=\frac{1}{2m^2}$; $b-3mc=0$, and $c=\frac{1}{2.3.m^3}$ &c.; $\therefore\ x=\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3.m^3}+$ &c., and $1+x=1+\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3.m^3}+$ &c. $=$ the number whose logarithm is $y$.

COR. If $m=1$, we obtain the number whose hyperbolic logarithm is $y$; it $=1+y+\frac{y^2}{2}+\frac{y^3}{2.3}+$ &c.

### (145.) PROB. 4.

To find the modular ratio.

By the last Proposition, the number whose logarithm is $y=1+\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3.m^3}+$ &c.; or $y$ is the measure of the ratio of this number to 1. Now the modular ratio is that ratio of which the modulus is the measure; if then $m=y$, $m$ will be the measure of this ratio, and the ratio itself becomes the modular ratio; that is, the modular ratio is the ratio of $1+1+\frac{1}{2}+\frac{1}{2.3}$ + &c. : 1; and therefore is the same for every system of logarithms, being independent both of $m$ and $y$.

(146.) By summing the series, it appears that this modular ratio is that of 2,7182818 &c. : 1. In NAPIER's system, where the modulus is 1, the logarithm of 2,7182818 is 1, The measure of this same ratio will be the modulus in any other system; the measure of that ratio in BRIGGS' system (where the logarithm of $10=1$), is ,4342[illegible] 9448, the modulus of his system.

(147.) In Art. 140. we assumed $y$ the measure of the ratio of $1+x : 1$, or the logarithm of $1+x$. If $y$ be the logarithm of $\frac{1}{1-x}$, or $-y$ the logarithm of $1-x$, then $-\dot{y} = \frac{-m\dot{x}}{1-x}$ and $\dot{y} = \frac{m\dot{x}}{1-x} = m \times : \dot{x}+x\dot{x}+x^2\dot{x}+x^3\dot{x}+$ &c.; and $y = m \times : x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+$ &c.

(148.) In the same manner, if in Art. 144. we take $y$ the logarithm of $\frac{1}{1-x}$, $\dot{y}-x\dot{y}-m\dot{x}=0$; and by assuming $x=ay+by^2+cy^3+$ &c. as before, we get $1-x=1-\frac{y}{m}+\frac{y^2}{2m^2}-\frac{y^3}{2.3m^3}+$ &c.

COR. 1. The modular ratio in this case = the ratio of $1 : 1-\frac{1}{1}+\frac{1}{2}-\frac{1}{2.3}+$ &c., or of 1 : ,3678794 &c.

COR. 2. Since $1+\frac{1}{1}+\frac{1}{2}+\frac{1}{2.3}+$ &c. $: 1 :: R+\frac{R}{1}+\frac{R}{2}+\frac{R}{2.3}+ : R$, if $\frac{R}{1}$ be assumed $=A$, $\frac{A}{2}=B$; $\frac{B}{3}=C$; $\frac{C}{4}=D$, &c. *ad infinitum*, and $S$ be put equal to $R+A+B+C+D+$ &c. the modular ratio will be that of $S : R$.

Or, since $1 : 1-\frac{1}{1}+\frac{1}{2}-\frac{1}{2.3}+$ &c. $:: S : S-\frac{S}{1}+\frac{S}{2}-\frac{S}{2.3}+$ &c. *ad infinitum*, if $\frac{S}{1}=A$, $\frac{A}{2}=B$, $\frac{B}{3}=C$, &c. and $R$ be taken equal to $S-A+B-C+$ &c. *ad infinitum*, the modular ratio is that of $S : R$. By the preceding cases, this ratio equal that of 2,7182818 &c. : 1, or of 1 : ,3678794, &c.

# CHAP. XIX.

## ON THE MAXIMA AND MINIMA OF CURVES.

### (149.) PROPOSITION.

To find the nature of curves in which some conditions being invariable, others become the greatest or the least possible.

The method of solving problems of this kind, will appear best by an example.

### (150.) EXAMPLES.

Ex. 1. Given the length of a curve, to find the area a maximum.

It is evident, that by merely putting the fluent of $y\dot{x}$ a maximum, no solution can be obtained; for no limitation is expressed, and the fluent will admit of increase without limit. But as the length is given, the $f.\ \dot{z}$, so far as concerns the $f.\ y\dot{z}$, is a given quantity; therefore the fluent of $y\dot{x} \pm f.\ \dot{z}$ must also be a maximum; or, to render the terms homogeneous, that they may admit of comparison, $f.\, y\dot{x} \pm f.\, a\dot{z}$ must be a maximum.

Now, if for every individual value of $y$, this flowing quantity be constantly a maximum, the whole fluent will be so; but for every such individual value of $y$, the flowing quantity is $y\dot{x} \pm a\dot{z}$. Hence, the nature of the curve will be determined by ascertaining, what relations of $\dot{x}$ and $\dot{z}$ will render $y\dot{x} \pm a\dot{z}$ a maximum, for any given value of $y$; or the fluxion of $y\dot{x} \pm a\dot{z}$ must $= 0$, whilst $y$ is constant; and this must be the case for every successive value of $y$ throughout; so that in each limiting portion of the area, for every value of $y$ the ratio of $\dot{x}$ and $\dot{z}$ must be such, as to make $y\dot{x} \pm a\dot{z}$ a maximum.

$\therefore\ y\ddot{x} \pm a\ddot{z} = 0$; but $\dot{z}^2 = \dot{y}^2 + \dot{x}^2$; $\therefore\ \dot{z}\ddot{z} = \dot{x}\ddot{x}$, and $\ddot{z} = \dfrac{\dot{x}\ddot{x}}{\dot{z}}$; $\therefore\ y\ddot{x} = \mp\dfrac{a\dot{x}\ddot{x}}{\dot{z}}$, and $y\dot{z} = \mp a\dot{x}$. But from the nature of the problem, $y\dot{z}$ must be positive; and therefore the true result is $y\dot{z} = a\dot{x}$.

Now in the circle, $a : y :: \dot{z} : \dot{x}$; $\therefore\ a\dot{x} = y\dot{z}$. Hence, the circle is the curve required; in which the length being given, the area is a maximum.

The same mode of reasoning may be adopted in the following cases. Hence the RULE.

(151.) If $A$ and $B$ denote any functions of $x$ and $y$, and $\dot{x} = \sqrt{c^2 \pm \dot{y}^2}$, where $c$ is constant, the expression $A\dot{x} \pm B\dot{y}$ is a maximum or minimum, when $A\dot{y} = \mp B\dot{x}$, or the functions of $x$ and $y$ are reciprocal.

Ex. 2. To determine the nature of a curve line, which generates a surface; so that the surface being given, the solid may be a maximum.

Here the $f.$ of $2py\dot{z}$ or of $y\dot{z}$ is given, and $f.\ y^2\dot{x}$ is a maximum; hence, the $f.\ ay\dot{z} \pm f.\ y^2\dot{x}$ is a maximum; $\therefore\ ay\dot{x} = y^2\dot{z}$ or $a\dot{x} = y\dot{z}$, a property of the circle; and the solid is a sphere.

Ex. 3. To determine the nature of the generating curve, that the solid being given, the surface may be a minimum.

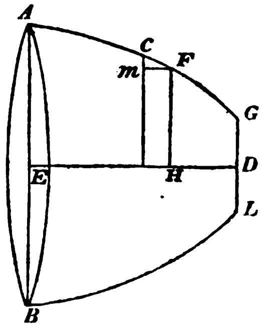
Here $f.\ y^2\dot{x}$ is given, and $f.\ ay\dot{z}$ is a minimum; $\therefore\ y^2\dot{z} = ay\dot{x}$, and $y\dot{z} = a\dot{x}$, a property of the circle.

Ex. 4. Given the length of the arc; to determine the nature of the curve, so that the solid may be a maximum.

Here $f.\ y^2\dot{x}$ is a maximum, and $f.\ \dot{z}$ or of $a^2\dot{z}$ is given; $\therefore f.\ y^2\dot{x} \pm f.\ a^2\dot{z}$ is a maximum, and $y^2\dot{z} = a^2\dot{x}$, the equation to the curve.

Since $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$, $y^4\dot{x}^2 + y^4\dot{y}^2 = a^4\dot{x}^2$, and $\dot{x} = \frac{y^2\dot{y}}{\sqrt{a^4 - y^4}}$, an equation in terms of $x$ and $y$.

Ex. 5. To find the nature of the curve which generates the solid of least resistance, when moving in a fluid, in the direction of its axis; its greatest ordinate *AB*, and length *DE*, being given.



If $DH = x$, $HF = y$, and $FC = \dot{z}$, the resistance on *CF* $\propto \frac{y\dot{y}^3}{\dot{z}^2}$, $\propto \frac{y\dot{y}^3}{\dot{z}^3} \times \dot{z}$, the fluent of which is a minimum. Also, the $f.\ a\dot{x}$ is given; $\therefore f.\ \frac{y\dot{y}^3}{\dot{z}^3} \times \dot{z} \pm f.\ a\dot{x}$ is a minimum; hence, $\frac{y\dot{y}^3}{\dot{z}^3} \times \dot{x} = a\dot{z}$, the equation to the curve.

Cor. The curve cannot meet the axis, for $y = \frac{a \times \dot{z}^4}{\dot{y}^3\dot{x}} = a \times \frac{CF^4}{CM^3 \times Fm}$, where the numerator is greater than the denominator; and therefore $y$ is greater than $a$.

Ex. 6. Given the area of the generating plane, and the greatest ordinate; to find the curve which generates the solid of least resistance.

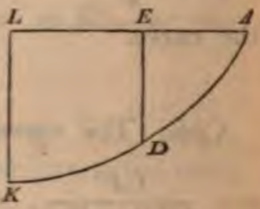
Here $f.\ ay\dot{x}$ is given, whilst $f.\ \frac{y\dot{y}^3}{\dot{z}^3} \times \dot{z}$ is a minimum; $\therefore f.\ \frac{y\dot{y}^3}{\dot{z}^3} \times \dot{z} \pm f.\ ay\dot{x}$ is a minimum; hence, $ay\dot{z} = \frac{y\dot{y}^3}{\dot{z}^4} \times \dot{x}$, and $a = \frac{\dot{y}^3\dot{x}}{\dot{z}^4}$, the equation to the curve.

Cor. Since $\frac{\dot{y}^3\dot{x}}{\dot{z}^4}$ is constant, the angle contained between the tangent *FC* and *Fm* is invariable; hence, the solid is either a cone, or the frustum of a cone.

Ex. 7. To find the same, when the greatest ordinate and content are given.

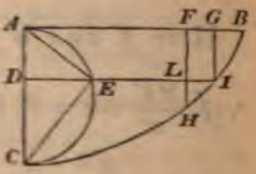
Here $f.\ \frac{y\dot{y}^3}{\dot{z}^3} \times \dot{z}$ is a minimum, and $f.\ y^2\dot{x}$ is given; or making the terms of the same dimensions, $f.\ \frac{ay\dot{y}^3}{\dot{z}^3} \times \dot{z} \pm f.\ y^2\dot{x}$ is a minimum; hence, $\frac{ay\dot{y}^3}{\dot{z}^3} \times \dot{x} = y^2\dot{z}$; and $a = \frac{y\dot{z}^4}{\dot{y}^3\dot{x}}$, the equation to the curve. The fluent is found in Ex. 20., on Fluxional Equations.

Ex. 8. To determine the nature of a curve *ADK*, down which a body will descend from *A* to *K* in the shortest time possible; the points *A* and *K* being given, and the velocity varying as the $m^{\text{th}}$ power of the ordinate.



Since $\dot{t} \propto \frac{\dot{z}}{v}$, where $\dot{z}$ is the fluxion of the curve, if $y = DE$, $\dot{t} \propto \frac{\dot{z}}{y^m}$. Now the points *A* and *K* are given; therefore $f.\ \frac{\dot{x}}{a^m}$ is given; and $f.\ \frac{\dot{z}}{y^m} \pm f.\ \frac{\dot{x}}{a^m}$ is a minimum. Hence, by the Rule, $y^m\dot{z} = a^m\dot{x}$, the equation to the curve.

Cor. 1. If the velocity ∝ as the square root of the ordinate, $m = \frac{1}{2}$, and $y^{\frac{1}{2}}\dot{z} = a^{\frac{1}{2}}\dot{x}$, which is a property of the cycloid.



For $IH : IL :: CE : ED :: CA : AE$

$:: \sqrt{CA} : \sqrt{AD}$

or, $\dot{z} : \dot{x} :: \sqrt{a} : \sqrt{y}$;

$\therefore\ y^{\frac{1}{2}}\dot{z} = a^{\frac{1}{2}}x.$

Cor. 2. If $m = 1$, $a\dot{x} = y\dot{z}$, which is the property of the circle.

Ex. 9. To determine the nature of the curve, down which a body will descend from an horizontal line *AL* to a vertical line $LK$, in the least time possible; the area being given, and the velocity varying as the $m^{th}$ power of the ordinate.

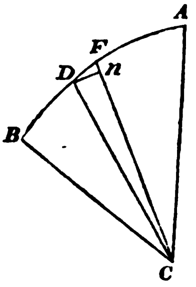
Here the $f.$ of $y\dot{x}$ is given; and since $\dot{t} \propto \frac{\dot{z}}{V} \propto \frac{\dot{z}}{y^m}$, $f. \frac{\dot{z}}{y^m}$ is a minimum; or $f. \frac{\dot{z}}{y^m} \pm f. \frac{y\dot{x}}{a^{m+1}}$ is a minimum; $\therefore \frac{y\dot{z}}{a^{m+1}} = \frac{\dot{x}}{y^m}$, or $y^{m+1}\dot{z} = a^{m+1}\dot{x}$, the equation to the curve.

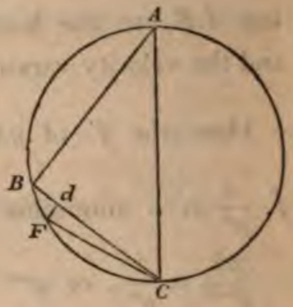
Cor. 1. Let $m = 0$; then $y\dot{z} = a\dot{x}$, which is an equation to the circle. Here the velocity is uniform.

Cor. 2. Let $m = \frac{1}{2}$, or the body fall by gravity; then, $y^{\frac{3}{2}}\dot{z} = a^{\frac{3}{2}}\dot{x}$; $\therefore y^3 \times \overline{\dot{x}^2 + \dot{y}^2} = a^3\dot{x}^2$; and by reduction $\dot{x} = \frac{y^{\frac{3}{2}}\dot{y}}{\sqrt{a^3 - y^3}}$, the equation to the curve.

Ex. 10. A body moving uniformly from one given point $A$ to another $B$, is impelled by a force tending to $C$, which $\propto \frac{1}{CD^2}$, where $CD$ is any variable intermediate distance; to determine the curve, so that its whole action upon the body, or $F \times \dot{t}$, may be a minimum.

Let $DCF$ be a small angle, $Dn = \dot{x}$, $CD = y$, $FD = \dot{z}$. Then, since the body moves uniformly, $\dot{T}$, which $\propto \frac{\dot{z}}{V} \propto \dot{z}$; $\therefore F \times \dot{T} \propto \frac{\dot{z}}{y^2}$ a minimum. Now the $f. \frac{\dot{x}}{y}$ is the measure of the angle $ACB$, and is therefore given; hence, $f. \frac{\dot{z}}{y^2} \pm f. \frac{\dot{x}}{ay}$ is a minimum; or $\frac{\dot{z}}{ay} = \frac{\dot{x}}{y^2}$, and $y\dot{z} = a\dot{x}$, a property of the circle, whose diameter is $a$.

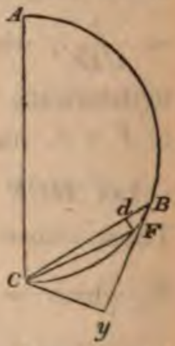




For if *CA* be a diameter, *CB* and *CF* two chords indefinitely near, and *FD* perpendicular to *CB*, the triangles *FBD*, *BAC*, are similar; and $BF : FD :: CA : CB$, or $\dot z : \dot x :: a : y$; $\therefore y\dot z = a\dot x$.

Ex. 11. To determine the curve, in which a body will move from a given point *A* to another *B* in the least time possible; the velocity being supposed to vary as $CD^m$.

Here $f.\,\dot t$, or $f.\,\frac{\dot z}{v}$, or $f.\,\frac{\dot z}{y^m}$ is a minimum; and the $f.\,\frac{Dn}{CD}$, or $f.\,\frac{\dot x}{y}$, is given; that is, multiplying the quantities $b$ and $a^m$ into the denominators of these expressions to make them homogeneous, $f.\,\frac{\dot z}{by^m} \pm f.\,\frac{\dot x}{a^m y} =$ a minimum; hence, $by^m\,\dot z = a^m y\dot x$, or $by^{m-1}\dot z = a^m\,\dot x$, the equation to the curve.



Cor. 1. If *ABC* be the curve, *CB* and *CF* be assumed indefinitely near, and *Cy* be drawn perpendicular on the tangent *By*; since $\frac{\dot x}{\dot z} = \frac{by^{m-1}}{a^m}$, we have $\frac{Fd}{FB}$ or $\frac{Cy}{CB}$ or $\frac{Cy}{y} = \frac{by^{m-1}}{a^m}$, and $Cy = \frac{by^m}{a^m}$.

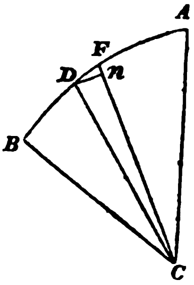
Cor. 2. If $m = 0$, or the velocity be constant, $Cy = b$, and the body describes a right line.

Cor. 3. If $m = 1$, $Cy = \frac{by}{a}$; or $CB : Cy :: a : b$ a given ratio, and the curve is the logarithmic spiral.

Cor. 4. If $m = 2$, the curve is a circle; for here $\frac{a^2}{b} \times$

$\dot{x} = y\dot{z}$; its diameter is $\frac{a^2}{b}$, and its circumference passes through *C.*

Ex. 12. Given *CA, CB,* and the length of the curve *ADF*; to find its nature, so that the area may be a maximum.



The fluxion of the area $= \frac{CD \times DN}{2} = \frac{y\dot{x}}{2}$; and $f.\,\dot{z}$, or of $a\dot{z}$, is given; $\therefore f.\,y\dot{x} \pm f.\,a\dot{z}$ is a maximum. Hence, $y\dot{z} = a\dot{x}$; and the curve is a circle, whose diameter is $a$.

# CHAP. XX.

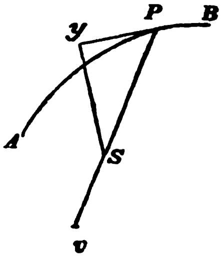
## ON THE APPLICATION OF FLUXIONS TO THE MOTIONS OF BODIES AFFECTED BY CENTRIPETAL FORCES.

## SECTION I.

### (152.) PROPOSITION.

To deduce a fluxional expression, for the variation of the force in any given curve, considered as a spiral.

If $S$ be the center of force corresponding to any orbit $AB$, $Pv$ the chord of curvature at $P$, and $Sy$ a perpendicular upon the tangent $Py$, the force by which a body would revolve in the orbit $AB$ varies as $\frac{1}{Sy^2 \times Pv}$. (NEWTON, Sect. 2. Prop. 6.)



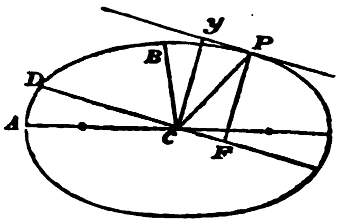
Let $SP = x$, $Sy = p$; then $Pv = \frac{2p\dot{x}}{\dot{p}}$ (Art. 118.);

$\therefore$ by substitution $F \propto \dfrac{1}{p^2 \times \dfrac{2p\dot{x}}{\dot{p}}} \propto \dfrac{\dot{p}}{p^3\dot{x}}$.

### (153.) EXAMPLES.

Ex. 1. To find the law of the force, by which a body may be made to describe an ellipse round the center.

Let $AC$ the semi-axis major $= a$, $CB$ the semi-axis minor $= b$, $CP = x$, $CD$ the semi-conjugate $= y$, and $Cy$ the perpendicular on the tangent $= PF = p$.



Then, by the nature of the curve, $PF^2 \times CD^2 = AC^2 \times BC^2$, or $\frac{1}{p^2} = \frac{y^2}{a^2b^2}$; $\therefore -\frac{\dot{p}}{p^3}$ varies as $y\dot{y}$. Also $x^2+y^2=a^2+b^2$; $\therefore x\dot{x}=-y\dot{y}$; $\therefore \frac{\dot{p}}{p^3}$ varies as $x\dot{x}$, and $\frac{\dot{p}}{p^3\dot{x}}$ varies as $x$, or the force varies as the distance from the center.

Cor. If the ellipse be changed into a parabola by the indefinite production of the axis major, then at any finite distance from the vertex $A$, $x$ is infinite; hence in this case the force is constant, and may be conceived to act in a direction parallel to the axis.

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Ex. 2. To find the law of the force, by which a body may be made to describe an hyperbola; the center of force being in the center of the figure.

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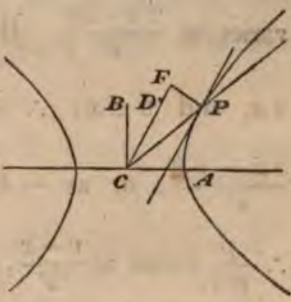
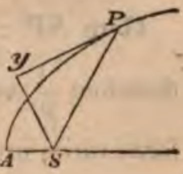


Here $PF^2 \times CD^2 = CA^2 \times CB^2$, or $\frac{1}{p^2} = \frac{y^2}{a^2b^2}$; $\therefore \frac{-\dot{p}}{p^3}$ varies as $y\dot{y}$, and $\frac{\dot{p}}{p^3\dot{x}} \propto \frac{-y\dot{y}}{\dot{x}}$. Also $x^2-y^2=a^2-b^2$; $\therefore x\dot{x}=y\dot{y}$; $\therefore \frac{\dot{p}}{p^3\dot{x}}$ varies as $-x$; or the force is repulsive, and proportional to the distance from the center.

Ex. 3. To find the law of the force in a parabola round the focus $S$.

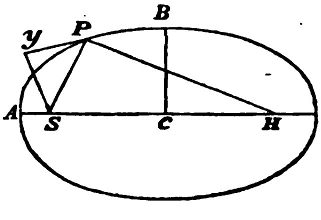
Sect III Prop. 13

Let $SP=x$, $Sy=p$, $SA=a$; then $p^2=ax$, or varies as $x$. $\therefore \frac{1}{p^2}$ is as $\frac{1}{x}$, and $\frac{\dot{p}}{p^3}$ as $\frac{\dot{x}}{x^2}$; $\therefore \frac{\dot{p}}{p^3\dot{x}}$ varies as $\frac{1}{x^2}$, or the force is inversely as $SP^2$.



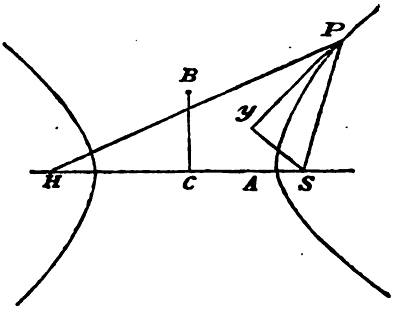
Ex. 4. To find the same in an ellipse round the focus. Sect. III. Prop. 11.

Let $AC=a$, $CB=b$, $SP=x$, $Sy=p$, $HP=v$. Then, by Conic Sections, $Sy^2=\frac{b^2\times x}{v}$; $\therefore \frac{1}{p^2}$ varies as $\frac{v}{x}$, and $\frac{-\dot{p}}{p^3}$ is as $\frac{x\dot{v}-v\dot{x}}{x^2}$. But $x+v=2a$, $\therefore \dot{v}=-\dot{x}$; and $\frac{-\dot{p}}{p^3}$ varies as $-\frac{x\dot{x}+v\dot{x}}{x^2}$; $\therefore \frac{\dot{p}}{p^3\dot{x}}$ varies as $\frac{x+v}{x^2}$, or as $\frac{2a}{x^2}$. Hence the force is as $\frac{1}{SP^2}$.



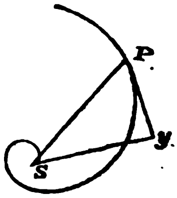
Ex. 5. To find the same in an hyperbola round the focus. Sect. III. Prop. 12.

Assuming as before, $p^2=\frac{b^2x}{v}$; $\therefore \frac{1}{p^2}$ varies as $\frac{v}{x}$, and $\frac{-\dot{p}}{p^3}$ varies as $\frac{\dot{v}x-\dot{x}v}{x^2}$. But $v-x=2a$, and $\dot{v}=\dot{x}$; $\therefore \frac{-\dot{p}}{p^3}$ is as $\frac{x\dot{x}-v\dot{x}}{x^2}$, or as $-\dot{x}\times\frac{\overline{v-x}}{x^2}$; $\therefore \frac{\dot{p}}{p^3\dot{x}}$ varies as $\frac{v-x}{x^2}$, or as $\frac{2a}{x^2}$; that is, the force varies inversely as $SP^2$.



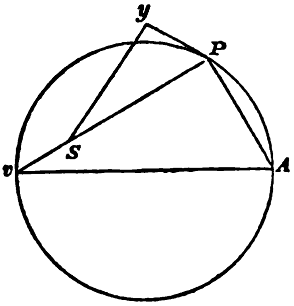
Ex. 6. To find the same, if the center of force be situate in the pole of a logarithmic spiral. Sect. II. Prop. 9.

Here $SP : Sy :: a : b$, or $x$ varies as $p$; therefore $\frac{1}{p^2}$ varies as $\frac{1}{x^2}$, and $\frac{\dot{p}}{p^3\dot{x}}$ as $\frac{1}{x^3}$; or the force varies inversely as $SP^3$.



Ex. 7. If the center of force be situated in a point $S$, which is not the center of the circle; to find the law of the force. Sect. II. Prop. 7. —

Suppose the body at $P$; join $PS$, and produce it to $v$; draw $vA$ a diameter, $Py$ a tangent, and $Sy$ a perpendicular on the tangent; join $AP$. Let $Av = a$, $SP = x$, $Sy = p$.

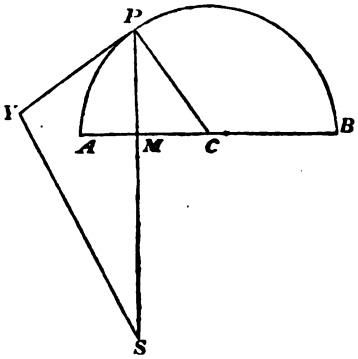


By similar triangles, $AvP$, $SPy$, $SP : Sy :: Av : Pv$; $\therefore Sy$ varies as $SP \times Pv$, or $p$ varies as $x \times \frac{2p\dot{x}}{\dot{p}}$; $\therefore \frac{\dot{p}}{\dot{x}}$ varies as $x$, and $\frac{\dot{p}}{p^3 \dot{x}}$ as $x \times \frac{1}{x^3 \times Pv^3}$; or the force varies as $\frac{1}{SP^2 \times Pv^3}$.

Cor. If the point $S$ coincide with $v$, or the center of force be in the circumference, $F \propto \frac{1}{Pv^5}$. Sect. II Prop. 7. Cor.

Ex. 8. To find the law of the force, by which a body may describe the semi-circle $APB$, the force acting in parallel lines, and perpendicular to $AB$. [illegible]

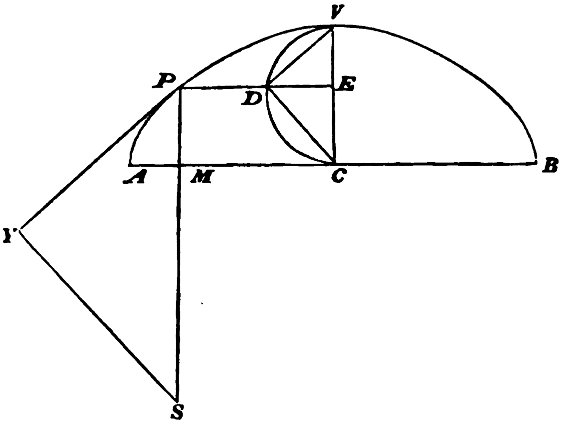
Draw $PM$ perpendicular to $AB$, and suppose it produced indefinitely to $S$; draw $PY$ a tangent, and let $SY$ be conceived perpendicular to $PY$; join $CP$. Then, by similar triangles, $CPM$, $SPY$, $SP^2 : SY^2 :: CP^2 : PM^2$; $\therefore SY^2$ varies as $PM^2$, since $SP^2$ is infinite; or, if $SY = p$, and $PM = y$, $\frac{1}{p^2} \propto \frac{1}{y^2}$; $\therefore \frac{\dot{p}}{p^3} \propto \frac{\dot{y}}{y^3}$; but $\dot{y} =$ the fluxion



of $SP = \dot{x}$; $\therefore \frac{\dot{p}}{p^3 \dot{x}}$ varies as $\frac{1}{y^3}$, or the force varies as $\frac{1}{PM^3}$.

Ex. 9. Let $APB$ be a cycloid, and the force act in parallel lines perpendicular to the base $AB$.

Let $V$ be the vertex, and $VDC$ the generating circle. Then

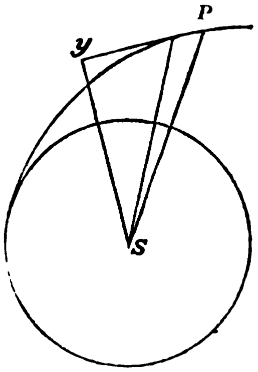


if $PDE$ be drawn parallel to $AB$, and $DV$, $DC$, be joined, $SP : SY :: CV : CD :: \sqrt{CV} : \sqrt{CE}$; $\therefore SY \propto \sqrt{\overline{CE}}$ $\propto \sqrt{y}$; hence $\frac{1}{p^2} \propto \frac{1}{y}$, and $\frac{\dot{p}}{p^3} \propto \frac{\dot{y}}{y^2} \propto \frac{\dot{x}}{y^2}$; $\therefore \frac{\dot{p}}{p^3 \dot{x}} \propto \frac{1}{y^2}$, and the force $\propto \frac{1}{PM^2}$.

Ex. 10. To find the law of the force in a spiral, whose equation is $p = \frac{ax}{\sqrt{a^2 + x^2}}$.

Here $\frac{1}{p^2} \propto \frac{a^2 + x^2}{x^2}$; $\therefore -\frac{2\dot{p}}{p^3} \propto \frac{2x^3\dot{x} - 2x\dot{x} \times \overline{a^2 + x^2}}{x^4} \propto$ $-\frac{2a^2 x\dot{x}}{x^4}$; $\therefore$ the force, or $\frac{\ddot{p}}{p^3 \dot{x}}$, $\propto \frac{1}{x^3}$.

Ex. 11. Let the curve be the involute of a circle, and the center of force in the center of the circle; to find the law.



Take $SP = x$, $Sy = p$; then, by the nature of the curve, $Py = r$, and $p = \sqrt{x^2 - r^2}$; $\therefore \frac{1}{p^2} = \frac{1}{x^4 - r^4}$; and $\frac{\dot{p}}{p^3\dot{x}} \propto \frac{x}{\overline{x^2 - r^2}|^2} \propto \frac{SP}{Sy^4}$.

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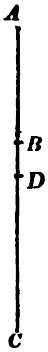
## SECTION II.

(154.) *On the motion of bodies in non-resisting mediums, when urged by forces tending to a center, and varying according to some power of the distance.*

### (155.) PROPOSITION.



If $v$ represent a velocity generated by a constant force $F$ through a space $x$, $v^2 = 4mFx$. Let $AB =$ the space through which the body has descended; let $BD = \dot{x}$, and $F$ represent the force at $B$. Then however $F$ may vary, yet through a small element of space $BD$, it may be considered as constant; therefore $2v\dot{v} = +4mF\dot{x}$, and $v\dot{v} = +2mF\dot{x}$. If $CB = x$, then $BD = -\dot{x}$; therefore in this case $2v\dot{v} = -4mF\dot{x}$; $\therefore v\dot{v} = -2mF\dot{x}$, whence in general $v\dot{v} = \pm 2mF\dot{x}$.

COR. $v\dot{v}$ varies as $\pm F\dot{x}$.

If $F = 1$, and $z$ represent the space through which a body must fall by this force to acquire the velocity $v$, $v^2 = 4mz$; $\therefore v\dot{v} = \pm 2m\dot{z}$;

$$\therefore 2m\dot{z} = \pm 2mF\dot{x}, \text{ and } \dot{z} = \pm F\dot{x}.$$

If $x$ decrease as the velocity increases, $\dot{z} = -F\dot{x}$. If $x$ increases with the increase of velocity, $\dot{z} = +F\dot{x}$; and $v = \sqrt{4mz} = \sqrt{4m} \times \sqrt{f.\ F\dot{x}}$.

### (156.) PROPOSITION.

Let $T =$ the time in which a body acquires the velocity $v$, when urged by the constant force $F$; then $v = 2mFT$; $\therefore$ for a small element of time as $\dot{T}$, $\dot{v} = 2mF\dot{T}$. But $S$ or $x = \frac{T \times V}{2}$;

$$\therefore \pm\dot{x} = \frac{T\dot{v} + v\dot{T}}{2} = \frac{2mFT\dot{T} + 2mFT\dot{T}}{2} = 2mFT\dot{T} = V \times \dot{T};$$

$\therefore \dot{T} = \frac{\pm\dot{x}}{V}$. If $x$ increase with the increase of the time, $\dot{T} = \frac{+\dot{x}}{v}$; if $x$ decrease as the time increases, $\dot{T} = \frac{-\dot{x}}{V}$.

### (157.) EXAMPLES.

Ex. 1. Suppose the force to vary as the $\overline{\text{distance}}|^{n-1}$ from the center; to find the variation of the velocity.

Here $v\dot{v} \propto -F\dot{x} \propto -x^{n-1}\dot{x}$; $\therefore v^2 \propto \frac{-x^n}{n} +$ corr. Let $v = 0$, when $x = a$; then $v^2 \propto a^n - x^n$, and $v \propto \sqrt{a^n - x^n}$.

Ex. 2. To find the variation of the time.

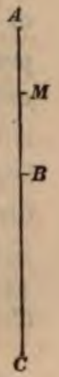
Here $\dot{T} \propto \frac{-\dot{x}}{\sqrt{a^n - x^n}}$, and the fluent gives the variation of $T$; but this fluent can only be found in particular cases.

Ex. 3. Let the force vary $\frac{1}{D^{n+1}}$, where $D$ is the distance from the center; to find the variation of the velocity.

Here $v\dot{v} \propto \frac{-\dot{x}}{x^{n+1}}$; $\therefore v^2 \propto \frac{1}{nx^n} +$ corr. Let $v = 0$, when $x = a$; then $v^2 \propto \frac{1}{x^n} - \frac{1}{a^n} \propto \frac{a^n - x^n}{x^n}$; $\therefore v \propto \sqrt{\frac{a^n - x^n}{x^n}}$.

Ex. 4. To find the variation of the time.

Here $\dot{T} \propto \frac{-x^{\frac{n}{2}}\dot{x}}{\sqrt{a^n - x^n}}$; of which the fluent must be found in particular cases.



Ex. 5. To find the actual velocity acquired by a body in falling through a space *AB*, when urged by a constant force *F*, situated in the point *C*.

Let $AC = a$, and $x$ = any variable distance; then $v\dot{v} = -2mF\dot{x}$, and $\frac{v^2}{2} = -2mFx +$ corr. When $v = 0$, $x = a$; $\therefore v^2 = 4mF \times \overline{a-x}$, and $v = \sqrt{4mF \times \overline{a-x}}$.

Cor. The velocity acquired through $AB \propto \sqrt{AB}$.

Ex. 6. To find the time of falling.

Here $\dot{T} = \frac{-\dot{x}}{V} = \frac{-\dot{x}}{\sqrt{4mF} \times \sqrt{a-x}} = \frac{1}{\sqrt{4mF}} - \dot{x} \times \overline{a-x}|^{-\frac{1}{2}}$,

$\therefore T = \frac{2}{\sqrt{4mF}} \times \overline{a-x}|^{\frac{1}{2}} +$ corr.; and when $x = a$, $T = 0$;

$\therefore$ corr. $= 0$. Hence, $T = \sqrt{\frac{1}{mF}} \times \overline{a-x}|^{\frac{1}{2}}$.

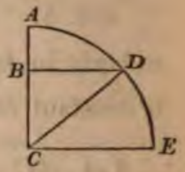
Cor. The time $\propto \sqrt{AB}$. Sect. VII. Prop. 38

Ex. 7. Let a body fall from rest at *A*, when urged by a force situated in *C*, which varies directly as the distance; to find the velocity acquired through *AB*.

Let the force at some point $M = F$; then, if $CM = r$, and $x =$ any variable distance, $r : x :: F$ : the force at $B = \frac{Fx}{r}$;

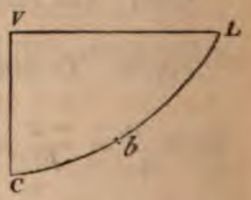
$\therefore v\dot{v} = -\frac{2mF}{r} \times x\dot{x}$, and $v^2 = \frac{2mF}{r} \times -x^2 +$ corr. $= \frac{2mF}{r} \times \overline{a^2 - x^2}$; $\therefore$ the velocity $= \sqrt{\frac{2mF}{r}} \times \sqrt{a^2 - x^2}$.

Cor. 1. If with $C$ as a center, and $CA$ radius, the quadrant of a circle be described, and $BD$ be drawn perpendicular to $AC$, the velocity at $B \propto BD$, $\propto$ the right sine of a circular arc, whose radius is the greatest distance from which the body begins to fall, and versed sine is the space through which it has descended.



Cor. 2. The velocity acquired at $B$ : that acquired at $C$ :: $BD : CE$ :: $BD : CD$.

Cor. 3. Since the force in a cycloid varies as the distance of the pendulum from the lowest point, if a pendulum be conceived to revolve in a semi-cycloid, $CL$ = the line $CA$, and the magnitude of the force at $L$ = that at $A$, its motion will correspond with that of a body descending through $AC$; and if $Cb = CB$, the velocity of the pendulum at $b$ must be as the right sine of a circular arc, whose radius $= CL$, and versed sine = the space $Lb$.



Ex. 8. The same things remaining, let it be required to find the time.

Here $\dot{T} = \frac{-\dot{x}}{V} = \sqrt{\frac{r}{2mF}} \times \frac{-\dot{x}}{\sqrt{a^2 - x^2}} = \sqrt{\frac{r}{2mFa^2}} \times \frac{-a\dot{x}}{\sqrt{a^2 - x^2}}$; $\therefore T = \sqrt{\frac{r}{2mFa^2}} \times$ a circular arc, whose radius $= a$, and cosine $x = \sqrt{\frac{r}{2mFa^2}} \times$ the arc $AD$. (Art. 44., Case 5.)

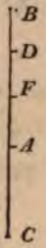
Cor. 1. The time $\propto$ as the arc $AD$; and $T, AB : T, AC :: AD : AE$.

Cor. 2. Since the arc of the quadrant $AE = \frac{pa}{2}$, where $p = 3.14159$, &c., the actual time of descent from $A$ to $C$ $= \sqrt{\frac{r}{2mFa^2}} \times \frac{pa}{2} = \frac{p}{2} \times \sqrt{\frac{r}{2mF}}$.

Cor. 3. In this expression $a$ is not found; hence, when the force varies as the distance from the lowest point, the times of descent from all altitudes to that point are the same, whether the bodies descend in straight lines, or in curves.

Cor. 4. The time of descent from $A$ to $C$ = half the time in which a pendulum, whose length is $r$, would oscillate by the action of the constant force $F$.

Ex. 9. Let the force vary inversely as the distance from $C$, and be repulsive; to find the velocity acquired through a given space $AB$, and the time of describing it.



Let the force at some point $F = F$; then if $CD = x$, $CF = r$, the force at $D = \frac{Fr}{x}$, and $v\dot{v} = +\frac{2mFr\dot{x}}{x}$; $\therefore v^2 = 4mFr \times$ hyp. log. $x +$ corr. Let $CA = a$, the velocity being nothing at $A$; then $v = \sqrt{4mFr \times \text{hyp. log.}\ \frac{x}{a}}$.

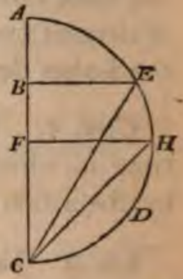
Also $\dot{T} = +\frac{\dot{x}}{v} = \frac{1}{\sqrt{4mFr}} \times \frac{\dot{x}}{\sqrt{\text{hyp. log.}\ \frac{x}{a}}}$; and

[redacted] $T = \frac{1}{\sqrt{4mFr}} \times : 2av^{\frac{1}{2}} + \frac{2av^{\frac{3}{2}}}{3.1} +$ &c., where $v =$ hyp. log. of $\frac{x}{a}$ (*fl.* 130.)

Ex. 10. Let the force vary as $\frac{1}{\overline{\text{dist.}}|^2}$ from $C$; to find the velocity and time.

Here the force at a variable distance $x = \frac{Fr^2}{x^2}$; $\therefore v\dot{v} = - 2mFr^2 \times \frac{\dot{x}}{x^2}$, and $v^2 = 4mFr^2\,\overline{\frac{1}{x} - \frac{1}{a}}$ when corrected, $= 4mFr^2 \times \overline{\frac{a-x}{ax}}$; $\therefore v = \sqrt{\frac{4mFr^2}{a}} \times \sqrt{\frac{a-x}{x}}$.

Also $\dot{T} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4mFr^2}} \times \frac{-x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}} = \sqrt{\frac{a}{4mFr^2}} \times \frac{-x\dot{x}}{\sqrt{ax-x^2}}$; and $T = \sqrt{\frac{a}{4mFr^2}} \times : \sqrt{ax-x^2}$ − a circular arc of radius $\frac{1}{2}a$ and versed sine $x$ (*fl.* 23.); that is, if upon $AC$ as a diameter a semi-circle be described, and the ordinate $BE$ be drawn, the time through $AB = \sqrt{\frac{a}{4mFr^2}} \times : BE - CDE + \text{corr.} = \sqrt{\frac{a}{4mFr^2}} \times : BE - CDE + CDA = \sqrt{\frac{a}{4mFr^2}} \times : BE + AE$.



Cor. 1. The velocity at $B \propto \sqrt{\frac{AB}{AC \times CB}}$; or in different parts of the same descent $\propto \sqrt{\frac{AB}{CB}}$.

Cor. 2. The velocity acquired in falling to the center is infinite.

Cor. 3. The time down $AB \propto$ the arc + the sine.

Cor. 4. If $p =$ the circumference of a circle, whose diameter is 1, $\frac{pa}{2} = ADC$, the semi-circumference to a diameter $a$; and the whole time of descent from $A$ to $C = \sqrt{\frac{a}{4mFr^2}} \times \frac{pa}{2} = \frac{p}{4r} \times \frac{a^{\frac{3}{2}}}{\sqrt{mF}}$.

**Cor. 5.** If $Q$ represent the length of a quadrant, $R$ the radius of the circle, and the force $\propto \frac{1}{D^2}$, the time of descent through the first half of a straight line toward the center : the time through the remaining half :: $Q + R : Q - R$.

Bisect $AC$ in $F$, and draw $FH$ perpendicular to $AC$; then, $T$ down $AF : T$, down $AC :: AH + HF : ADC$;

$$\therefore T, AF : T, FC :: AH + HF : ADC - AH - HF$$
$$:: Q + R : Q - R.$$

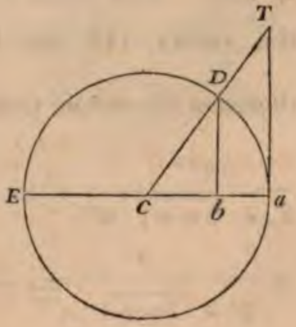
**Ex. 11.** Let the force tending to $C$ vary inversely as the cube of the distance; to find the velocity and time.

In this case, the force at a variable distance $x = \frac{Fr^3}{x^3}$;

$\therefore v\dot{v} = -2mFr^3 \times \frac{\dot{x}}{x^3}$; $\therefore v^2 = 2mFr^3 \times \frac{1}{x^2} + \text{corr.} = 2mFr^3 \times \overline{\frac{1}{x^2} - \frac{1}{a^2}} = 2mFr^3 \times \frac{a^2 - x^2}{a^2x^2}$; $\therefore v = \sqrt{2mFr^3} \times \frac{\sqrt{a^2 - x^2}}{ax}$.

Also $\dot{T} = \frac{a}{\sqrt{2mFr^3}} \times \frac{-x\dot{x}}{\sqrt{a^2 - x^2}}$; and $T = \frac{a}{\sqrt{2mFr^3}} \times \sqrt{a^2 - x^2}$.

**Cor. 1.** With $C$ as a center, and $Ca = CA$ as radius, describe a circle $aDE$. Take $ab = AB$; draw $bD$ perpendicular to $aC$, and let $aT$ be the tangent of $aD$. Then $Cb : bD :: Ca : aT$, or $x : \sqrt{a^2 - x^2} :: a : aT$; $\therefore aT \propto \frac{\sqrt{a^2 - x^2}}{x}$; but $v \propto \frac{\sqrt{a^2 - x^2}}{x}$; $\therefore$ the velocity acquired through $AB \propto$ as the tangent of a circular arc, whose radius

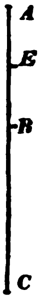


is *CA*, and versed sine the space through which the body has descended.

Cor. 2. The velocity acquired at *C* is infinite.

Cor. 3. The time through $AB \propto Db \propto$ the right sine of of the arc, whose versed sine is the space described.

Ex. 12. If the force $\propto D^{n-1}$, required an expression for the whole time of descent to the center.



In this case, the force at a distance $x = \frac{Fx^{n-1}}{r^{n-1}}$; $\therefore v\dot{v} = \frac{-2mF}{r^{n-1}} \times x^{n-1}\dot{x}$; and $v^2 = \frac{4mF}{nr^{n-1}} \times \overline{a^n - x^n}$; $\therefore v = \sqrt{\frac{4mF}{nr^{n-1}}} \times \sqrt{a^n - x^n}$.

Hence $\dot{T} = \frac{-\dot{x}}{v} = \sqrt{\frac{nr^{n-1}}{4mF}} \times \frac{-\dot{x}}{\sqrt{a^n - x^n}} = \sqrt{\frac{nr^{n-1}}{4mFa^n}} \times -\dot{x} \times \overline{1 - \frac{x^n}{a^n}}\Big|^{-\frac{1}{2}} =$ (by the Binomial Theorem) $\sqrt{\frac{nr^{n-1}}{4mFa^n}} \times : -\dot{x} - \frac{x^n\dot{x}}{2a^n} - \frac{3x^{2n}\dot{x}}{2.4.a^{2n}} - \frac{3.5.x^{3n}\dot{x}}{2.4.6.a^{3n}} -$ &c.; $\therefore T = \sqrt{\frac{nr^{n-1}}{4mFa^n}} \times : -x - \frac{x^{n+1}}{2.\overline{n+1}.a^n} - \frac{3x^{2n+1}}{2.4.\overline{2n+1}.a^{2n}} -$ &c. $+$ corr. Now when $T=0$, $x=a$; and when the body reaches the center, all the terms which contain $x$ must vanish; therefore the whole time $= \sqrt{\frac{nr^{n-1}}{4mFa^n}} \times : a + \frac{a^{n+1}}{2.\overline{n+1}.a^n} + \frac{3a^{2n+1}}{2.4.\overline{2n+1}.a^{2n}} +$ &c. $= \sqrt{\frac{nr^{n-1}}{4mF}} \times : \frac{1}{a^{\frac{n-2}{2}}} + \frac{1}{2.\overline{n+1}.a^{\frac{n-2}{2}}} + \frac{3}{2.4.\overline{2n+1}.a^{\frac{n-2}{2}}} +$ &c.

COR. The whole time $\propto \frac{1}{a^{\frac{n-2}{2}}}$.

EXAMPLES.

1. Let the force be constant; $n-1=0$; $\therefore n=1$, and $T \propto \sqrt{a}$.

2. Let $F \propto$ the distance : $n-1=1$; $\therefore n=2$; $\therefore T \propto \frac{1}{a^0} \propto 1$, or is constant from all altitudes.

3. Let $F \propto \frac{1}{D^2}$; $n-1=-2$; $\therefore n=-1$; and $T \propto a^{\frac{3}{2}}$.

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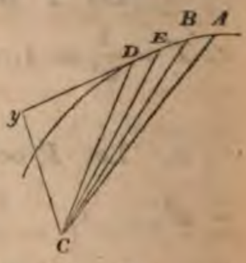
Ex. 13. If the attractive force in $C$ vary as $\frac{1}{D^{n+1}}$, and a body be projected from $B$ in the direction $BA$, with a given velocity; required to find the height to which it will rise. (See the FIGURE in the preceding page.)

Let $CA$ the height required $=p$, $CB=r$, $CE$ a variable distance $=x$; let $c=$ the velocity of projection, and $v$ the velocity at $E$. Let the force at $B=F$. Then the force at $E=\frac{Fr^{n+1}}{x^{n+1}}$; $\therefore v\dot{v}=-\frac{2mFr^{n+1}\dot{x}}{x^{n+1}}$; and $v^2=\frac{4mFr^{n+1}}{nx^n}+$ corr. When $v=c$, $x=r$; $\therefore v^2-c^2=\frac{4mFr^{n+1}}{nx^n}-\frac{4mFr}{n}$. Let $v=0$, or the body arrive at $A$; $x=p$, and $c^2=\frac{4mFr}{n}-\frac{4mFr^{n+1}}{np^n}$; $\therefore np^nc^2=4mFrp^n-4mFr^{n+1}$. Hence, $p^n \times \overline{4mFr-nc^2}=4mFr^{n+1}$; $\therefore p=r \times \overline{\frac{4mFr}{4mFr-nc^2}}\Big|^{\frac{1}{n}}$.

## SECTION III.

### (158.) Proposition.

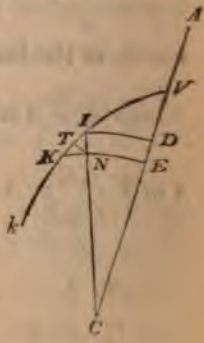
If a body describe any curve round a center of force, the velocity at any point varies inversely as the perpendicular drawn from that center upon the tangent.



Let $C$ be the center of force; $AB$, $BE$, $ED$, small parts of the curve described in equal successive portions of time. On $Ey$, a tangent at $E$, draw the perpendicular $Cy$; then $CED$ may be considered as a small rectilinear triangle, and its area $CED \propto DE \times Cy$; $\therefore DE \propto \frac{CED}{Cy}$. But the arc $DE$ described in a small given time is proportional to the velocity at $E$; and (by Newton, Prop. 1.) all the areas $ACB$, $BCE$, $ECD$, &c. described round $C$, *dato tempore*, are equal; $\therefore v \propto \frac{1}{Cy}$.

### (159.) Proposition.

Let $C$ represent a center of force, and suppose one body to describe round that center a curve $VIK$, whilst another descends in a right line by the action of the same force toward $C$; if their velocities are equal in one instance, when at equal distances from $C$, in the curve and the straight line, they will be equal at all other equal distances.



Let $IK$ represent a small part of the curve; with $C$ as a center, and radii $CI$, $CK$, describe the circular arcs $ID$, $KE$; draw $NT$ perpendicular to $IK$; and suppose that the velocity of the body descending from $A$ to $C$, when at $D$, to equal the velocity in the curve at $I$. Let $DE$ or $IN$ represent the force at $D$ or $I$ toward the center; $IT$ will represent that part of it which impels the body in the curve. Now the times in which $DE$ and $IK$ are described, since the velocities are equal at $D$ and $I$, will be proportional to $DE$ and $IK$; and the increment of the velocity is as the force and the increment of the time, or $\dot{v} \propto F \times \dot{T}$;

But $F$ in the curve : $F$ in the line :: $IT : DE$,
and $T$ in the curve through $IK$ : $T$ through $DE$ :: $IK : DE$.

Hence ($F \times \dot{T}$ or) $\dot{v}$ in the curve : ($f. \times \dot{t}$) or $\dot{v}$ in the line :: $IT \times IK : DE^2$.

But since $INK$ is a right angle, $IT \times IK = DE^2$; $\therefore$ the increments of velocity through $IK$ and $DE$ are equal. In the same manner, the corresponding increments, and therefore the whole velocities at equal distances, will continue to be equal.

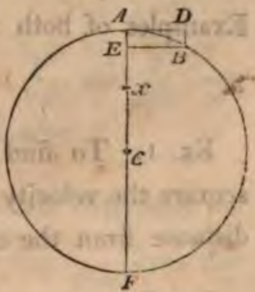
Cor. 1. If the body at $D$ projected upward with the velocity at $D$ would ascend to $A$ before it loses all its velocity, the body at $I$, projected from $I$ in the direction $CI$ produced, would rise to a distance equal to $CA$.

Cor. 2. Hence, since $v\dot{v} \propto \pm F\dot{x}$ in the line, the same expression obtains in the curve.

### (160.) Proposition.

If a body revolve uniformly in a circle round the center $C$, the velocity of the body is equal to that which it would acquire by falling through one-fourth of the diameter, when urged by the constant force in the circumference.

Take $AD$ a tangent at $A$, very small. Draw $DB$ parallel to $AC$, and $BE$ perpendicular to it; and suppose $Ax$ to be the space through which a body must fall to acquire the velocity at $A$. Now whilst the body describes $AB$ with an uniform velocity, another would fall through $DB$ by the constant force at $A$, and would afterwards describe $2DB$ in the same time with the velocity at $B$ continued uniform. Hence, the



vel. of the body at $A$ : vel. acquired thro' $DB$ :: $AB : 2DB$
and vel. thro' $DB$ : vel. acquired thro' $Ax$ :: $\sqrt{DB} : \sqrt{Ax}$;

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$\therefore$ vel. at $A$ : vel. acquired through $Ax$ :: $AB : 2\sqrt{DB \times Ax}$;
$\therefore$ by the hypothesis $AB = 2 \times \sqrt{DB \times Ax}$,

and $Ax = \dfrac{AB^2}{4DB} = \dfrac{AB^2}{4AE} = \dfrac{AF}{4}$; or a body must fall through a space $= \dfrac{AF}{4}$ when urged by the constant force at $A$, to acquire the velocity in the circle.

Cor. 1. If $m = 16\frac{1}{12}$, and $r =$ radius, the velocity at $A = \sqrt{\dfrac{4mr}{2}} = \sqrt{2mr} =$ the space uniformly described in 1″, the force at $A$ being assumed $= 1$.

Cor. 2. If $c =$ the circumference of a circle to the radius $r$, and $p = 3.14159$ &c. $=$ the circumference of a circle to the diameter 1, $c = 2pr$; hence, to find the whole time of revolution, we have $\sqrt{2mr} : 2pr :: 1'' : T''$;

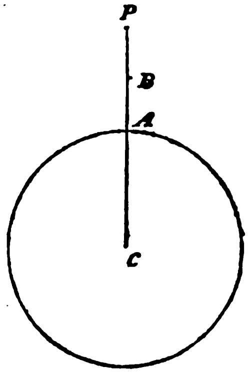
$$\therefore \text{ the time } = \frac{2pr}{\sqrt{2mr}} = p \times \sqrt{\frac{2r}{m}}.$$

In the following Problems, the velocities are determined from the equation $\dot{z} = \pm F\dot{x}$. It might have been applied in the cases preceding, but it was thought expedient to give Examples of both methods.

## Examples.

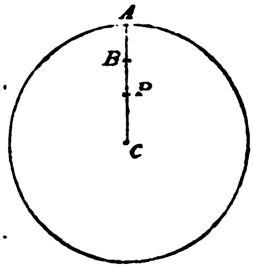
Ex. 1. To find how far a body must fall externally to acquire the velocity in a circle, the force varying directly as the distance from the center.

Let $P$ be the point from which the body must fall, $CB$ any



variable distance; suppose the force at $A = 1$, and $CA = a$, $CP = p$, $CB = x$. The force at $B = \frac{x}{a}$; ∴ if $z$ represent the space through which the body must fall by the constant force at $A$ to acquire the proper velocity, $\dot{z} = \frac{-x\dot{x}}{a}$, and $z = \frac{-x^2}{2a}$ + corr. Let $z = 0$, $V = 0$; ∴ $x = cP = p$; hence, $z = \frac{p^2 - x^2}{2a}$; and when $x = cA$, or the body comes to $A$, $z = \frac{p^2}{2a} - \frac{a}{2}$. But $z = \frac{a}{2}$ (by Art. 160.); ∴ $\frac{p^2}{2a} - \frac{a}{2} = \frac{a}{2}$; hence, $p^2 = 2a^2$, and $p$ or $cP = a \times \sqrt{2}$.

Ex. 2. To find how far a body must fall internally to acquire the velocity in a circle, the force varying directly as the distance from the center.



Suppose $P$ the point to which it must descend, and let $CB = x$ a variable distance. The force at $B = \frac{x}{a}$; ∴ $\dot{z} = \frac{-x\dot{x}}{a}$, and $z = \frac{-x^2}{2a}$ + corr. Let $x = a$; then $v = 0$, and therefore $z = 0$. Hence, $C = \frac{a^2}{2a} = \frac{a}{2}$; ∴ $z = \frac{a}{2} - \frac{x^2}{2a}$, and when $x = p$, or the body comes down to $P$, $z = \frac{a}{2} - \frac{p^2}{2a}$. But $z = \frac{a}{2}$; ∴ $\frac{a}{2} = \frac{a}{2} - \frac{p^2}{2a}$, and $\frac{p^2}{2a} = 0$, that is, $p = 0$; ∴ the body must fall to the center.

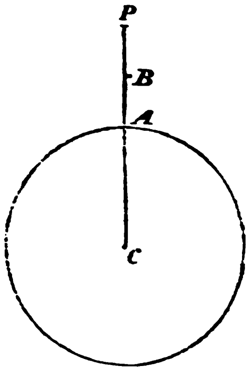
Ex. 3. To find how far a body must fall externally to acquire the velocity in a circle, the force varying $\frac{1}{\text{dist.}}$ from the center.

The force at $B=\frac{a}{x}$; $\therefore\ \dot{z}=\frac{-a\dot{x}}{x}$, and $z=-a\times$ hyp. log. $x+$ corr. $=a\times$ hyp. log. $\frac{p}{x}$, when the body comes to $a=$ $a\times$ hyp. log. $\frac{p}{a}$; $\therefore\ \frac{a}{2}=a\times$ hyp. log. $\frac{p}{a}$; $\therefore$ hyp. log. $\frac{p}{a}=\frac{1}{2}$. Let $e=2.71828$ &c. $=$ the number, whose hyperbolic logarithm is 1; then (Art. 136. Cor.) $e^{\frac{1}{2}}=\frac{p}{a}$; $\therefore p=a\times e^{\frac{1}{2}}=cP$.

Ex. 4. To find the same internally.

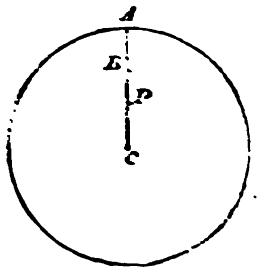
The force at $B$ within the circle $=\frac{a}{x}$; $\therefore\ \dot{z}=\frac{-a\dot{x}}{x}$, and $z=-a\times$ hyp. log. $x+$ corr. $=a\times$ hyp. log. $\frac{a}{x}$ (when the body comes to $P$) $a\times$ hyp. log. $\frac{a}{p}$; that is, $\frac{a}{2}=a\times$ hyp. log. $\frac{a}{p}$. Hence, assuming $e$ as before, $p=\frac{a}{e^{\frac{1}{2}}}=cP$.

Ex. 5. To find how far a body must fall externally to acquire the velocity in a circle, if the force vary as $\frac{1}{D^2}$ from the center.



The force at $B=\frac{a^2}{x^2}$; $\therefore\ \dot{z}=-\frac{a^2\dot{x}}{x^2}$, and $z=\frac{a^2}{x}-\frac{a^2}{p}$. When the body comes to $A$, it $=a-\frac{a^2}{p}$; that is, $\frac{a}{2}=a-\frac{a^2}{p}$; $\therefore\ \frac{a^2}{p}=\frac{a}{2}$; $\therefore p$ or $cP=2a$.

Ex. 6. To find the same internally.



The force at $B=\frac{a^2}{x^2}$; $\therefore\ \dot{z}=-\frac{a^2\dot{x}}{x^2}$, and $z=\frac{a^2}{x}-\frac{a^2}{a}$. Let the body come to $P$, and $\frac{a}{2}=\frac{a^2}{p}-a$; $\therefore\ \frac{a^2}{p}=\frac{3a}{2}$, and $p=\frac{2a}{3}$.

Ex. 7. To find the same externally, if the force $\propto \frac{1}{D^3}$.

The force at $B = \frac{a^3}{x^3}$; $\therefore \dot{z} = -\frac{a^3\dot{x}}{x^3}$, and $z = \frac{a^3}{2x^2} - \frac{a^3}{2p^2}$; or, when the body comes to $P$, $\frac{a}{2} = \frac{a}{2} - \frac{a^3}{2p^2}$; $\therefore p^2 = \frac{a^3}{0}$, or the body must fall from infinity to acquire the velocity at $A$.

Ex. 8. To find the same internally.

The force at $B = \frac{a^3}{x^3}$; $\therefore \dot{z} = -\frac{a^3\dot{x}}{x^3}$, and $z = \frac{a^3}{2x^2} - \frac{a}{2}$. Let the body come to $P$, then $\frac{a}{2} = \frac{a^3}{2p^2} - \frac{a}{2}$; $\therefore \frac{a^3}{2} = p^2 \times a$, and $p = \frac{a}{\sqrt{2}}$.

Ex. 9. To find how far a body must fall externally to acquire the velocity in a circle, the force varying as $\frac{1}{D^{n+1}}$ from the center.

The force at $B = \frac{a^{n+1}}{x^{n+1}}$; $\therefore \dot{z} = -\frac{a^{n+1}\dot{x}}{x^{n+1}}$, and $z = \frac{a^{n+1}}{nx^n} - \frac{a^{n+1}}{np^n}$; or, when the body comes to $A$, $\frac{a}{2} = \frac{a}{n} - \frac{a^{n+1}}{np^n}$; $\therefore \frac{a^{n+1}}{np^n} = \frac{a \times \overline{2-n}}{2n}$, and $p = a \times \sqrt[n]{\frac{2}{2-n}}$.

Ex. If $n = 1$, $F \propto \frac{1}{D^2}$, and $p = 2a$, as before.

If $n = 2$, $F \propto \frac{1}{D^3}$, and $p = a \times \sqrt{\frac{2}{0}}$ = an infinite quantity.

Ex. 10. How far must a body fall internally, to acquire the velocity in a circle, the force varying $\frac{1}{D^{n+1}}$.

Here $\dot{z} = \frac{-a^{n+1}\dot{x}}{x^{n+1}}$, and $z = \frac{a^{n+1}}{nx^n} - \frac{a}{n}$. When the body comes to $P$, $\frac{a}{2} = \frac{a^{n+1}}{np^n} - \frac{a}{n}$, or $\frac{a^{n+1}}{np^n} = \frac{\overline{n+2}\,.\,a}{2n}$; $\therefore p = a \times \sqrt[n]{\frac{2}{2+n}}$.

Ex. If $n = -2$, $F \propto$ dist. and $p = a \times \left.\frac{2}{0}\right]^{-\frac{1}{2}} = 0$, as before.

If $n = 1$, $p = \frac{2a}{3}$, and $F \propto \frac{1}{D^2}$.

If $n = 2$, $p = a \times \sqrt{\frac{1}{2}}$, and $F \propto \frac{1}{D^3}$.

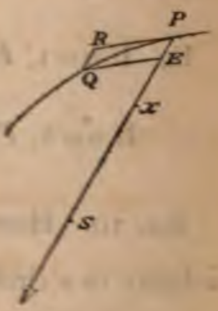
If the force vary as $D^{n-1}$, general expressions may be deduced in the same manner.

The distance $p$, if the body fall externally to acquire the velocity in a circle, $= a \times \sqrt{\frac{2+n}{2}}$. If it fall internally, $p = a \times \sqrt[n]{\frac{2-n}{2}}$.

In all these cases, we have considered the force situated in $C$ as attractive. If the force be repulsive, the same reasoning may be adopted to find the velocities and times of bodies ascending; only in this case, $2v\dot{v} = +4mF\dot{x}$, and $\dot{z} = +F\dot{x}$, for the velocity increases with the increase of $x$.

### (161.) LEMMA.

The space through which a body must fall to acquire the velocity in any curve at any point $P$, is equal to $\frac{1}{4}$th the chord of curvature at that point; the force of acceleration being equal to the force at $P$.



Let $PV$ be the chord of curvature passing through $S$ the center of force; $PQ$ a small arc, $PR$ a tangent at $P$, $QR$ parallel to $PV$, and $QE$ parallel to $PR$. Then, if $Px$ be the space through which the body falls by the constant force at $P$ to acquire the velocity at $P$, by the same process as in Art. 160. we have the

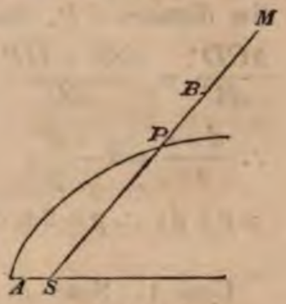
vel. with which $PQ$ is described : vel. acquired thro' $RQ$ :: $PQ : 2RQ$,
and vel. acquired through $RQ$ : vel. through $Px$ :: $\sqrt{RQ} : \sqrt{Px}$;

$\therefore$ vel. with which $PQ$ is described : vel. thro' $Px$ :: $PQ : 2\sqrt{RQ \times Px}$;

$\therefore$ since the first velocity = the second by hypothesis, $PQ = 2\sqrt{RQ \times Px}$; or $Px = \dfrac{PQ^2}{4QR} = \dfrac{PV}{4}$.

Cor. If the accelerating force be the same in two cases, the velocity $\propto$ the $\sqrt{\text{chord of curvature}}$; if the force be not the same, the velocity $\propto$ as the $\sqrt{F \times \text{chord of curvature}}$.

Ex. 11. How far must a body fall externally to acquire the velocity at any point $P$ in the parabola, $F$ varying $\dfrac{1}{D^2}$.



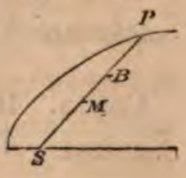
Let $c =$ the chord of curvature $= 4SP$, $SM = p$, $SB = x$, $SP = a$.

The force at $B = \dfrac{a^2}{x^2}$, if the force at $P = 1$; $\therefore \dot{z} = -\dfrac{a^2\dot{x}}{x^2}$, and $z = \dfrac{a^2}{x} - \dfrac{a^2}{p}$;

$\therefore$ when the body comes to $P$, $\dfrac{c}{4}$ or $a = a - \dfrac{a^2}{p}$; $\therefore \dfrac{a^2}{p} = 0$, or $p$ is infinite.

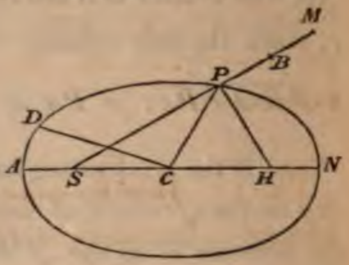
Ex. 12. How far must a body fall internally, the force varying $\dfrac{1}{D^2}$, to acquire the velocity in a parabola.

Here $\dot{z} = -\dfrac{a^2\dot{x}}{x^2}$; $\therefore z = \dfrac{a^2}{x} +$ corr. $= \dfrac{a^2}{x} - a$, and when the body comes to $M$, $\dfrac{c}{4}$ or $a = \dfrac{a^2}{p} - a$;



$\therefore 2p = a$, and $p = \dfrac{a}{2}$.

Ex. 13. How far must a body fall externally to acquire the velocity in an ellipse; the force tending to the focus, and varying $\frac{1}{D^2}$.

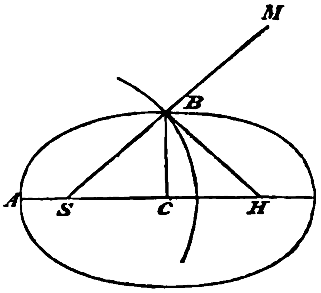


Let $M$ be the point from which the body must fall; assume $AN=2a$, $SM=p$, $SB=x$, $SP=d$, and $PH=v$. Then, if the force at $P=1$, the force at $B=\frac{d^2}{x^2}$; $\therefore\ \dot{z}=-\frac{d^2\dot{x}}{x^2}$, and $z=\frac{d^2}{x}-\frac{d^2}{p}$; and when the body comes to $P$, $\frac{c}{4}=d-\frac{d^2}{p}$. Now if $CD$ be the semi-conjugate diameter to the distance $CP$, the chord of curvature passing through $S=\frac{2CD^2}{AC}=\frac{2SP\times HP}{AC}$ (by Conics); $\therefore\ \frac{c}{4}=\frac{SP\times HP}{2AC}=\frac{d\times v}{2a}$; $\therefore\ \frac{d\times v}{2a}=d-\frac{d^2}{p}$; $\therefore\ \frac{d^2}{p}=\frac{2ad-dv}{2a}$; or $2a\times d=p\times\overline{2a-v}$ $=p\times d$; $\therefore p=2a$, or $SM=2AC=SP+HP$; $\therefore\ MP=HP$.

Cor. 1. Since in whatever part of the orbit $P$ is assumed, the line $SM$ always $=2a$; if with $S$ as a center, and a radius $=AN$, a circle be described, a body descending from any point in the circumference of that circle, in a line $MPS$ directly toward $S$, acquires a velocity at $P$ equal to the velocity of a body revolving in the ellipse.

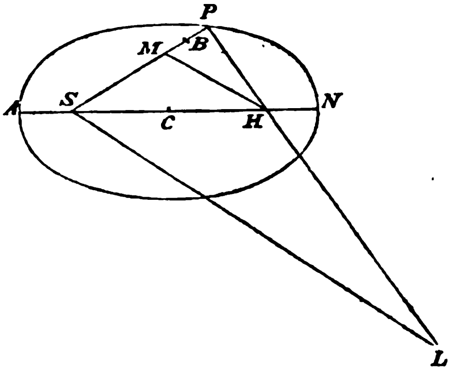
Cor. 2. A body projected from $P$ in the direction $SPM$, with the velocity in the ellipse at $P$, will just rise to the circumference of that circle before it loses all its velocity.

Cor. 3. If $BC$ be the semi-minor axis, and a circle be described with the center $S$, and radius $SB$, the velocity of a body revolving in that circle is equal to the velocity in the ellipse at *B*. For a body must fall through $MB = HB = SB$ to acquire the velocity in the ellipse at *B*; and (Art. 160. Ex 5.) it falls through the same space to acquire the velocity in the circle, whose radius is *SB*.



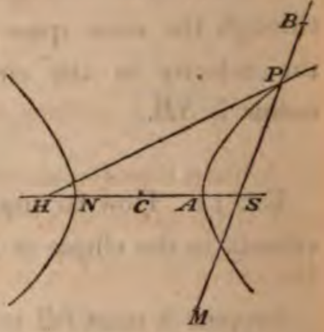
Ex. 14. How far must a body fall internally to acquire the velocity in the ellipse at *P*, under the same circumstances?

Suppose it must fall to *M*. Let $SM = p$, the force at $B = \frac{d^2}{x^2}$; $\therefore \dot{\mathfrak{z}} = \frac{-d^2\dot{x}}{x^2}$, and $\mathfrak{z} = \frac{d^2}{x} - d$. When the body comes to *M*, $\frac{c}{4} = \frac{d^2}{p} - d$, or $\frac{d \times v}{2a} = \frac{d^2}{p} - d$; $\therefore d \times \frac{\overline{2a+v}}{2a} = \frac{d^2}{p}$, and $p = \frac{2ad}{2a+v} = SM$; $\therefore PM = SP - SM = d - \frac{2ad}{2a+v} = \frac{d \times v}{2a+v} = \frac{SP \times HP}{AN + HP}$.



Cor. Since $PM = \frac{SP \times HP}{AN + HP}$; $\therefore PM : HP :: SP : AN + HP$. Hence, if *PH* be produced to *L*, and $HL = AN$, and *HM* be drawn parallel to *SL*, *PM* is that part of *SP* through which a body must fall to acquire the velocity at *P*. For $PM : HP :: SP : PL$.

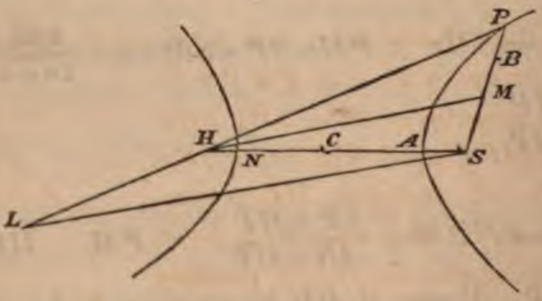
Ex. 15. How far must a body fall externally to acquire the velocity in an hyperbola; the force tending to the focus, and varying $\frac{1}{D^2}$?



The same suppositions being made as before, the force at $B = \frac{d^2}{x^2}$; $\therefore\ z = \frac{d^2}{x} - \frac{d^2}{p}$; and when the body comes to $P$, $\frac{c}{4} = d - \frac{d^2}{p}$; that is, $\frac{d \times v}{2a} = d - \frac{d^2}{p}$; $\therefore\ \frac{d^2}{p} = d - \frac{d \times v}{2a} = \frac{d \times \overline{2a - v}}{2a} = \frac{d \times -d}{2a}$; $\therefore\ p = -2a$. That is, if $PS$ be produced to $M$, and $SM$ be taken $= 2a$, the body by ascending from $M$ to an infinite distance, the force being considered repulsive, and then descending by the attractive force to $P$, will acquire the velocity at $P$ in the curve.

Ex. 16. How far must a body fall internally to acquire the velocity in the hyperbola at $P$, the force varying $\frac{1}{D^2}$?

The force at $B = \frac{d^2}{x^2}$; $\therefore\ z = \frac{d^2}{x} - d$; and when the body comes to $M$, $\frac{c}{4}$ or $\frac{d \times v}{2a} = \frac{d^2}{p} - d$; $\therefore\ \frac{d^2}{p} = \frac{\overline{2a+v} \times d}{2a}$, or $p = \frac{2a \times d}{2a+v}$; $\therefore\ PM = d - \frac{2a \times d}{2a+v} = \frac{d \times v}{2a+v} = \frac{HP \times SP}{AN + HP}$.

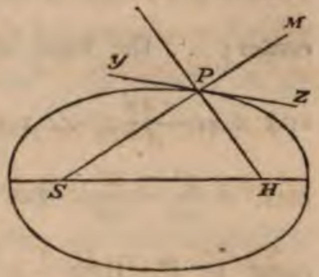


Cor. If in $PH$ produced, $HL$ be taken $= AN$, $SL$ be joined, and $HM$ drawn parallel to it, it cuts off from $PS$ a part $PM$ through which a body must fall to acquire the velocity in the hyperbola at $P$.

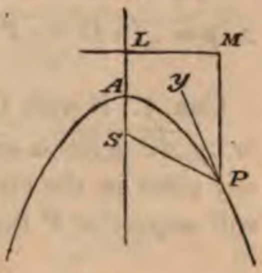
## Proposition.

### *Corollary to the Three last Cases.*

A body is projected at a given angle, with a given velocity, and at the distance $SP$ from the center of force; to determine the curve in which it will move, the force in $S$ varying $\frac{1}{SP^2}$.

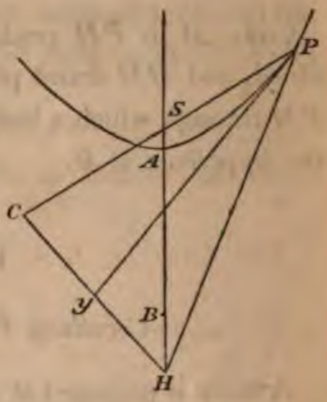


1. Let the velocity be equal to that acquired in falling through a finite distance $MP$. Let $Py$ be the direction of projection; produce $yP$ to $z$; join $SP$. Make the angle $zPH$ equal to $yPS$; take $PH=PM$; and with $S$, $H$ as foci, and $SP+PH$ as major axis, describe an ellipse. That ellipse is the curve required.



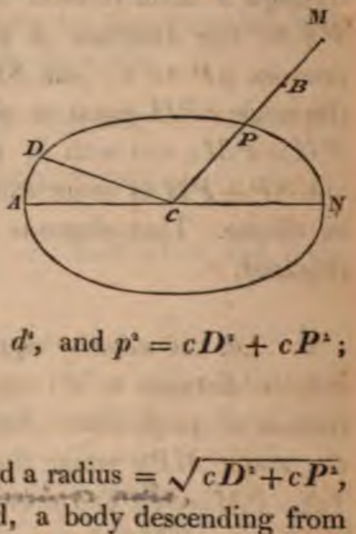
2. Let the velocity equal that acquired in falling from an infinite distance to $P$, and $Py$ the direction of projection. Join $SP$; make the angle $MPy$ equal the angle $SPy$. Take $PM=SP$, and draw $ML$ perpendicular to $PM$; then with $S$ as focus, and $ML$ directrix, describe a parabola. That parabola is the curve required.

3. Let the velocity equal that acquired by ascending from some point $C$ to infinity by a repulsive force, and then descending



by the attractive force to *P*. Let *Py* be the direction of projection; make the angle $yPH$ = the angle $yPC$; draw $CyH$ perpendicular to $Py$; then with *S* and *H* as foci, and an axis major $AB = SC$, describe the hyperbola *PA*, concave toward the center of force. This hyperbola is the curve required.

Ex. 17. How far must a body fall externally to acquire the velocity in an ellipse, the center of force being in the center of the ellipse?

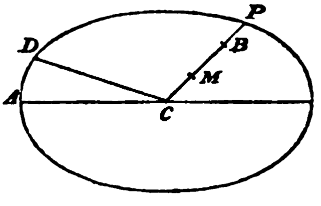


Here the force varies directly as the distance from the center; $\therefore$ the force at $B = \frac{x}{d}$, and $\dot{z} = \frac{-x\dot{x}}{d}$; $\therefore z = \frac{-x^2}{2d} +$ corr. $= \frac{p^2}{2d} - \frac{x^2}{2d}$. Let the body come to *P*, then $\frac{c}{4} = \frac{p^2}{2d} - \frac{d}{2}$; or $\frac{cD^2}{2d} = \frac{p^2}{2d} - \frac{d}{2}$; $\therefore cD^2 = p^2 - d^2$, and $p^2 = cD^2 + cP^2$; $\therefore p = \sqrt{cD^2 + cP^2}$.

Cor. 1. If with *C* as a center, and a radius $= \sqrt{cD^2 + cP^2}$, or $\sqrt{Ac^2 + Bc^2}$ where BC is the [illegible] radius, a circle be described, a body descending from any point in the circumference of that circle in a line *MPc*, will acquire at *P* the velocity in the ellipse at that point.

Cor. 2. Since with the same variation of the force a body falls externally through a space $= cP \times \sqrt{2}$, to acquire the velocity in a circle of radius $cP$ (Art. 160. Ex. 1.), the velocity in this circle and in the ellipse at $P$ will be equal, when $\sqrt{cD^2+cP^2}=cP\times\sqrt{2}$, or when $cD=cP$.

Ex. 18. How far must a body fall internally, under the same circumstances, to acquire the velocity in the ellipse at $P$?

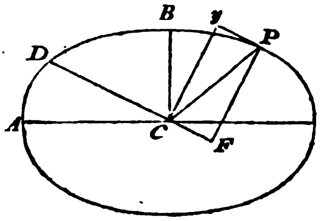


The force at $B=\frac{x}{d}$, $\therefore\ \dot{z}=\frac{-x\dot{x}}{d}$, and $z=\frac{-x^2}{2d}+\frac{d}{2}$; $\therefore$ when the body comes to $M$, $\frac{c}{4}$ or $\frac{cD^2}{2d}=\frac{d}{2}-\frac{p^2}{2d}$; hence $p^2=cP^2-cD^2$; $\therefore\ p=\sqrt{cP^2-cD^2}$.

Cor. 1. If $cD=cP$, $p=0$. In this case, the body falls to the center; but it must fall to the center to acquire the velocity in a circle of radius $cP$ (Art. 160. Ex. 2.); whence it appears also, as in the last Case, that if $cD=cP$, the velocity in an ellipse at $P$ is equal to that in a circle at the same distance.

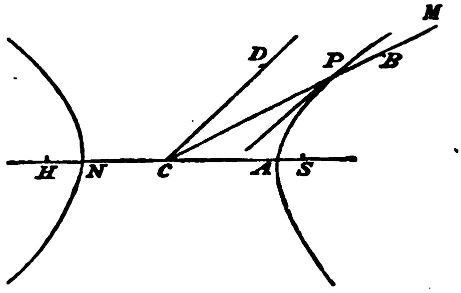
Cor. 2. If $cP$ is less than $cD$, $p$ becomes impossible; that is, the body by falling to $c$ cannot acquire the velocity in the ellipse at $P$.

Ex. 19. If a body be projected obliquely with the velocity acquired through a finite space $d$, by a force which varies as the distance from the center, it will describe an ellipse.



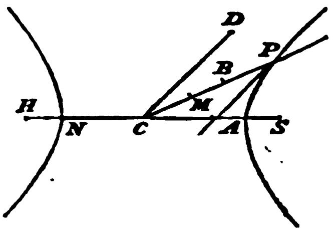
For $v\dot{v}\propto-F\dot{x}\propto-x\dot{x}$ (Art. 159. Cor. 2.); $\therefore\ v^2\propto d^2-x^2$, and $v\propto\sqrt{d^2-x^2}$. But (Art. 158.) $v\propto\frac{1}{cy}$; $\therefore\ cy\propto\frac{1}{\sqrt{d^2-x^2}}$, which is a property of the ellipse, where $d^2=AC^2+BC^2$.

Ex. 20. A body revolves in the hyperbola $AP$ by a repulsive force varying as the distance, of which the center is in the center of the hyperbola. To find how far it must rise from $P$, in the direction $CPM$, to acquire the velocity in the hyperbola at $P$.



Here $\dot{z} = +F\dot{x} = \frac{x\dot{x}}{d}$; $\therefore z = \frac{x^2}{2d} - \frac{d^2}{2d}$; that is, when the body comes to $M$, $\frac{cD^2}{2d} = \frac{p^2 - d^2}{2d}$; $\therefore p^2 = cD^2 + d^2$, and $p = \sqrt{cD^2 + cP^2} = cM$.

Ex. 21. The same supposition being made; to find from what point $M$ a body must rise to $P$, to acquire the velocity in the hyperbola at $P$.

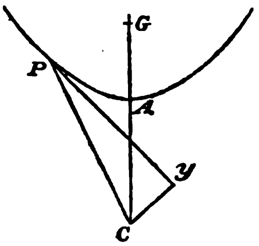


Here $\dot{z} = \frac{x\dot{x}}{d}$, and $z = \frac{x^2}{2d} - \frac{p^2}{2d}$. Let the body come to $P$, then $z = \frac{d^2 - p^2}{2d}$, or $\frac{cD^2}{2d} = \frac{d^2 - p^2}{2d}$; $\therefore p = \sqrt{cP^2 - cD^2} = cM$.

Cor. If $cP = cD$, or the hyperbola be rectangular, $cM = 0$.

Ex. 22. To determine the orbit described by a body, if projected from $A$ in a direction perpendicular to $AC$, with such a velocity as would be destroyed by descending to the center $C$; the force being repulsive, and varying directly as the distance from $C$.

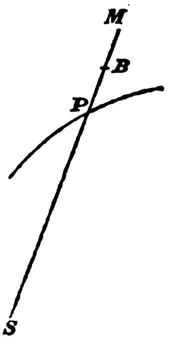
Suppose a body to ascend from $A$ in the line $CAG$ with the velocity in the curve at the point $A$; then, by Art. 159., the velocities in the curve and the line are equal at all other equal distances from the center of force. Now in the line, $v\dot{v} \propto F\dot{x} \propto x\dot{x}$; $\therefore v^2 \propto x^2$, and $v \propto x$, for the fluent needs no correction. Hence, if $CP = x$, the velocity in the curve at $P \propto$ as $CP$. But if $Cy$ be a perpendicular on the tangent $Py$, the velocity at $P \propto$ as $\frac{1}{Cy}$ (Art. 158.); $\therefore CP \propto$ as $\frac{1}{Cy}$, which is a property of the rectangular hyperbola.



Ex. 23. Universally, the force varying as $\frac{1}{D^{n+1}}$; to find how far a body must fall externally, to acquire the velocity in any curve at a given point $P$.

Let $M$ be the point required; take $SM = p$, $SB$ a variable distance $= x$, $SP = d$, and $c =$ the chord of curvature at $P$. Then the force at $P$ being assumed $= 1$, $\dot{z} = \frac{-d^{n+1}\dot{x}}{x^{n+1}}$; $\therefore z = \frac{d^{n+1}}{nx^n} - \frac{d^{n+1}}{np^n}$; or, when the body comes to $P$, $\frac{c}{4} = \frac{d}{n} - \frac{d^{n+1}}{np^n}$;

$$\therefore \frac{d^{n+1}}{np^n} = \frac{d - \frac{1}{4}nc}{n}, \text{ and } p = \frac{d \times d^{\frac{1}{n}}}{\sqrt[n]{d - \frac{1}{4}nc}} = SM.$$



### EXAMPLES.

1. Suppose the curve a circle, and $F \propto$ distance; $c = 2d$, and $n = -2$; $\therefore SM = d \times d^{-\frac{1}{2}} \times \sqrt{d+d} = \frac{d \times \sqrt{2d}}{\sqrt{d}} = d \times \sqrt{2}$.

2. Let $F \propto \frac{1}{D^2}$, the curve being a circle; here $n + 1 = 2$; $\therefore n = 1$, and $SM = \frac{d \times d}{d - \frac{d}{2}} = 2d$.

3. Let $F \propto \frac{1}{D^3}$, $n+1=3$; $\therefore n=2$, and $SM = \frac{d \times d^{\frac{1}{2}}}{\sqrt{d-d}}$ = an infinite quantity.

4. If the curve be a logarithmic spiral, $c=2d$; and the $F \propto \frac{1}{D^3}$; $\therefore SM = \frac{d \times d^{\frac{1}{2}}}{\sqrt{d-d}}$ = an infinite quantity.

5. In a parabola, $c = 4d$; and $F \propto \frac{1}{D^2}$, or $n = 1$; $\therefore SM = \frac{d \times d}{d-d}$ = an infinite quantity.

6. If the curve be a circle, and the center of force in the circumference, or $F \propto \frac{1}{D^5}$, $n+1=5$; $\therefore n=4$, and $c=d$; hence, $SM = \frac{d \times d^{\frac{1}{4}}}{\sqrt[4]{d-d}}$ = an infinite quantity.

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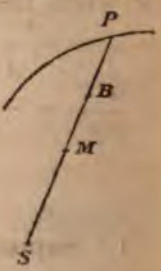
Ex. 24. The force varying as $\frac{1}{D^{n+1}}$; to find how far a body must fall internally, to acquire the velocity in any curve at a given point $B$.

Here $\dot{z} = \frac{-d^{n+1}\dot{x}}{x^{n+1}}$; $\therefore z = \frac{d^{n+1}}{nx^n} - \frac{d}{n}$; or, when the body comes to $M$, $\frac{c}{4} = \frac{d^{n+1}}{np^n} - \frac{d}{n}$; and $\frac{d^{n+1}}{np^n} = \frac{d+\frac{1}{4}nc}{n}$;

$$\therefore p = \frac{d \times d^{\frac{1}{n}}}{\sqrt[n]{d+\frac{1}{4}nc}} = SM.$$



### Examples.

Ex. 1. Suppose the curve a circle, and that the force varies as the distance; $c=2d$, and $n+1=-1$; $\therefore n=-2$; and $p = d \times d^{-\frac{1}{2}} \times \sqrt{d-d} = 0$, or the body falls to the center.

2. Let $F \propto \frac{1}{D^2}$, $n+1=2$; $\therefore n=1$, and $p = \frac{d^2}{d+\frac{d}{2}} = \frac{2d}{3}$.

3. Let $F \propto \frac{1}{D^3}$, $n=2$; $\therefore p = \frac{d \times \sqrt{d}}{\sqrt{d+d}} = \frac{d}{\sqrt{2}}$.

4. In the logarithmic spiral, $c = 2d$; and $F \propto \frac{1}{D^3}$; $\therefore n=2$, and $p = \frac{d \times \sqrt{d}}{\sqrt{d+d}} = \frac{d}{\sqrt{2}}$.

5. In the parabola, $F$ varies as $\frac{1}{D^2}$; $\therefore n=1$, and $c=4d$; $\therefore p = \frac{d^2}{d+d} = \frac{d}{2}$.

6. In a circle, which has the center of force in the circumference, $n+1=5$; $\therefore n=4$, and $c=d$; hence, $p = \frac{d \times d^{\frac{1}{4}}}{\sqrt[4]{d+d}} = \frac{d}{\overline{2}|^{\frac{1}{4}}}$.

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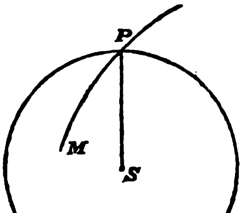
## SECTION IV.

### (162.) PROPOSITION.

To compare the velocity in a curve at any point with the velocity in a circle at the same distance from the center of force.

The velocity in the curve at $P$, is equal to that which a body would acquire in falling through one-fourth of the chord of curvature at that point, if urged by the constant force at $P$; and the velocity in the circle at $P$ is equal to that acquired through one-fourth of its diameter by the same force; hence, $V^2$ varies as the chords of curvature; or $V^2$ in the curve at $P$ : $V^2$ in the circle $SP$ :: $\frac{2p\dot{x}}{\dot{p}} : 2x$



(Art. 118.) :: $\frac{\dot{x}}{x} : \frac{\dot{p}}{p}$.

## EXAMPLES.

Ex. 1. To compare the velocity in a logarithmic spiral with the velocity in a circle at the same distance $SP$.

Here $x : p :: a : b$; $\therefore x : p :: \dot{x} : \dot{p}$, and $\frac{\dot{x}}{x} = \frac{\dot{p}}{p}$; hence the velocity in the spiral is equal to that in a circle at the same distance.

Ex. 2. To compare the same in a parabola.

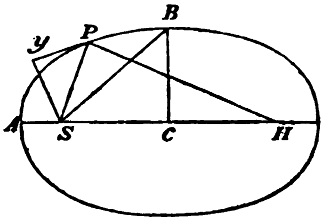
In this case, $p^2 = ax$; $\therefore 2p\dot{p} = a\dot{x}$; also $x = \frac{p^2}{a}$; $\therefore \frac{\dot{x}}{x} = \frac{2ap\dot{p}}{ap^2} = \frac{2\dot{p}}{p}$; $\therefore V^2$ in the parabola : $V^2$ in the circle $:: \frac{2\dot{p}}{p} : \frac{\dot{p}}{p} :: 2 : 1$; and $V$ in the parabola : $V$ in the circle at the same distance $:: \sqrt{2} : 1$.

Ex. 3. To compare the same in an ellipse round the focus.

Here $Sy^2 = BC^2 \times \frac{SP}{HP}$; or, if $AC = a$, $BC = b$, $p^2 = \frac{b^2 x}{2a - x}$;

$$\therefore 2p\dot{p} = \frac{2ab^2\dot{x}}{\overline{2a-x}|^2} = \frac{b^2 x}{2a-x} \times \frac{2a\dot{x}}{x \times \overline{2a-x}} = \frac{p^2 \times 2a\dot{x}}{x \times \overline{2a-x}}.$$

Hence,



$$\frac{\dot{x}}{x} = \frac{\dot{p}}{p} \times \frac{2a-x}{a};$$

$\therefore V^2$ in the ellipse at $P$ : $V^2$ in the circle $SP :: \frac{\dot{p}}{p} \times \frac{2a-x}{a} : \frac{\dot{p}}{p}$;

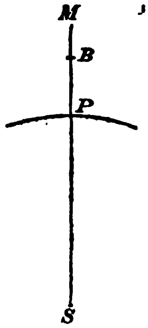
$:: HP : AC$;

and $V$ in the ellipse at $P$ : $V$ in the circle $SP :: \sqrt{HP} : \sqrt{AC}$.

Cor. The velocity at $B$ in the ellipse is equal to the velocity in a circle whose radius is $SB$.

In this and some other cases, it is more convenient to determine the ratio of the velocities by comparing the chords of curvature. For $V^2$ varies as the chord of curvature, when the accelerating force is the same. (Art. 161. Cor.)

In the two following Propositions, the velocities are compared by determining their actual values.



Ex. 4. To compare the velocity acquired by a body in falling from a given point *M* through a finite distance *MP*, with the velocity of a body in a circle at the distance *SP*, the force varying as $\frac{1}{D^{n+1}}$.

Let $SP=d$, and the force at $P=1$; $SM=p$, $SB$ a variable distance $=x$. Then the force at $B=\frac{d^{n+1}}{x^{n+1}}$; $\therefore\ \dot{z}=-\frac{d^{n+1}\dot{x}}{x^{n+1}}$, and $z=\frac{d^{n+1}}{nx^n}-\frac{d^{n+1}}{np^n}=\frac{d^{n+1}\times:p^n-x^n}{np^nx^n}$. But $V^2=4mz$; $\therefore V^2$ at $P$, where $x=d$, $=4md\times:\frac{p^n-d^n}{np^n}$.

Also $V^2$ in a circle at the distance $SP=4m\times\frac{d}{2}$;

$$\therefore \text{vel. of body falling at } P : \text{vel. in cir. } SP :: \sqrt{\frac{4md\times\overline{p^n-d^n}}{np^n}} : \sqrt{\frac{4md}{2}}$$

$$:: \sqrt{p^n-d^n} : \sqrt{\frac{np^n}{2}}.$$

Cor. 1. When the velocity of the falling body is equal to the velocity in a circle at that distance, $p^n-d^n=\frac{np^n}{2}$, and $p$ may be found; thus, if the force $\propto\frac{1}{D^2}$, since $n=1$, $p=2d$.

Cor. 2. If $F\propto\frac{1}{D^2}$,

$$\text{vel. of body falling in } P : \text{vel. in circle } SP :: \sqrt{p-d} : \sqrt{\frac{p}{2}}$$

$$:: \sqrt{MP} : \sqrt{\tfrac{1}{2}MS}.$$

(Newton, Sect. 7. Prop. 33.)

Ex. 5. The force varying according to the same law; to compare the velocity of a body falling from an infinite distance to a given point *P*, with the velocity in a circle at the distance *SP*.

By proceeding as in the last case, $z = \frac{d^{n+1}}{nx^n}$, and the correction $=0$; $\therefore$ when the body comes down to *P*, $z = \frac{d}{n}$;

hence, vel. of the falling body at *P* : vel. in cir. *SP* $:: \sqrt{\frac{4md}{n}} : \sqrt{\frac{4md}{2}}$

$:: \sqrt{2} : \sqrt{n}$.

### Examples.

Ex. 1. Let $F \propto \frac{1}{D^2}$, $n=1$; therefore the

vel. from infinity : vel. in a circle *SP* $:: \sqrt{2} : 1$,

as in the parabola.

Ex. 2. Let $F \propto \frac{1}{D^3}$, $n=2$; therefore the velocity from infinity is equal to the velocity in a circle *SP*, as in the logarithmic spiral.

Cor. 1. If the ratio of the velocity acquired from infinity at *P* to that in a circle be given $:: c : 1$, the law of the force may be found. For $c : 1 :: \sqrt{2} : \sqrt{n}$; $\therefore n+1 = \frac{2+c^2}{c^2}$.

Cor. 2. If a body be projected from *P* in the direction *PM*, with a velocity which is to the velocity in a circle at *P* $:: \sqrt{2} : \sqrt{n}$, it will go off *ad infinitum*; and this is the least velocity with which it can be projected, so that it may never return.

Cor. 3. Since $V^2 \propto$ as the chord of curvature when the force is the same, we can compare the chord of curvature of the curve described by a body projected perpendicularly at $P$; its velocity being that acquired from infinity, with the given distance $SP$.

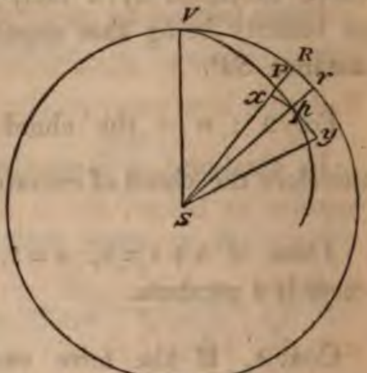
For $2 : n :: $ the chord of curvature required $: 2SP$; therefore the chord of curvature $= \frac{4SP}{n}$.

Thus, if $n+1=2$, $n=1$, and the chord $= 4SP$; or the curve is a parabola.

Cor. 4. If the force vary inversely as the cube of the distance, $n=2$; and a body projected at $P$, in a direction perpendicular to $SP$ with the velocity acquired from infinity, will in this case describe a circle. If the force vary in a higher inverse ratio than $\frac{1}{D^3}$, 2 is less than $n$; hence, the velocity of the body is less than that in a circle at the same distance; but $V^2 \propto PV$, where the force is given. Hence, the chord of curvature of the curve described is less than that of a circle, whose radius is $SP$; and the body will describe an orbit interior to the circle, and at last fall into the center. If the force vary in a less inverse ratio than $\frac{1}{D^3}$, 2 is greater than $n$, or the velocity of the body is greater than that in a circle of radius $SP$; therefore, since $V^2 \propto PV$, the chord of curvature of the curve described is greater than the diameter of the circle; hence, the body will describe an orbit exterior to the circle, and at length go off to infinity. The two following Problems are connected with this subject.

### (163.) Prob. 1.

If the force vary as $\frac{1}{D^n}$ from the center, where $n$ is greater than 3, and a body be projected from $V$ in a direction perpendicular to the line $SV$, with the velocity acquired in falling from an infinite distance, it is required to find after how many revolutions it will fall into the center.



Let $VPp$ be the curve described; $VRr$ a circle, whose radius is $SV$; SPR & $SPr$ two lines drawn from $S$ indefinitely near to each other. Draw the tangent $Py$, $Sy$ perpendicular to it, and $px$ perpendicular to $SP$. Take $SP=x$, $SV=r$. Then in the curve (Art. 159.) $v\dot{v} \propto -F\dot{x} \propto \frac{-\dot{x}}{x^n}$;

$\therefore V^2 \propto \frac{1}{x^{n-1}}$, and the correction vanishes. But, by Art. 158. $V \propto$ inversely as the perpendicular.

Hence, $V^2$ at $V : V^2$ at $P :: x^{n-1} : r^{n-1} :: Sy^2 : SV^2$;
$\therefore Sy^2 = \frac{r^2 x^{n-1}}{r^{n-1}} = \frac{x^{n-1}}{r^{n-3}}$; $\therefore Py^2 = SP^2 - Sy^2 = x^2 - \frac{x^{n-1}}{r^{n-3}} = \frac{x^2 r^{n-3} - x^{n-1}}{r^{n-3}}$. Now, by similar triangles, $Ppx$, $SPy$,

$$Px : px :: Py : Sy;$$

and $px : Rr :: SP : SR$;

$$\therefore Px : Rr :: SP \times Py : Sy \times SR;$$ or, if $VR=z$,

$$-\dot{x} : \dot{z} :: \sqrt{x^4 r^{n-3} - x^{n+1}} : \sqrt{r^2 x^{n-1}};$$

$\therefore \dot{z} = \frac{-rx^{\frac{n-1}{2}}\dot{x}}{x^2\sqrt{r^{n-3}-x^{n-3}}} = \frac{-rx^{\frac{n-5}{2}}\dot{x}}{\sqrt{r^{n-3}-x^{n-3}}}$; whose fluent (*f.* 48.)

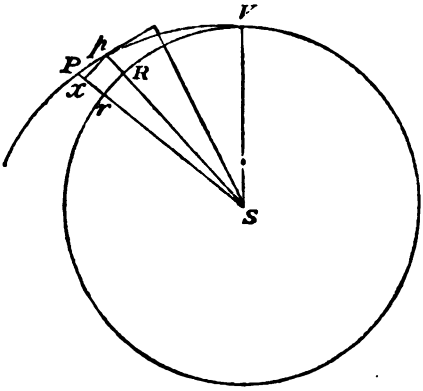
or $z = \frac{2r}{\overline{n-3}.r^{\frac{n-3}{2}}} \times$ a circular arc of radius $= r^{\frac{n-3}{2}}$, and cosine $x^{\frac{n-3}{2}}$, which needs no correction. Let $x=0$, and $r=1$; then $z = \frac{2}{n-3} \times$ a quadrant $= \frac{2}{n-3} \times \frac{\text{circumference}}{4} = \frac{\text{circumference}}{2n-6}$; that is, the body describes $\frac{1}{2n-6}$ revolutions, before it falls into the center.

Cor. If $n=4$, the body falls into the center after half a revolution.

### (164.) PROB. 2.

If the force $\propto \frac{1}{D^n}$ from the center, where $n$ is greater than 1 and less than 3, and a body be projected as in the last case; to find after how many revolutions it will go off *ad infinitum.*

In this case, $\dot{x}$ is positive; and by reasoning in the same manner as before, $Rr$ or $\dot{z}$ is found $= \frac{rx^{\frac{n-3}{2}}\dot{x}}{\sqrt{r^{n-3} - x^{n-3}}}$; and the fluent ($f.$ 48.) $= -\frac{\times 2r}{\overline{n-3}\,.\,r^{\frac{n-3}{2}}} \times$ a circular arc of radius $= r^{\frac{n-3}{2}}$, and cosine $x^{\frac{n-3}{2}}$, which needs no correction. When $r$ is unity, and $x$ is infinite (or when the cosine $=0$, since $n$ is less than 3, and therefore $\frac{n-3}{2}$ negative), it $= \frac{2}{3-n} \times$ a quadrant $= \frac{1}{6-2n}$ revolutions.



COR. If $n = 1$, $\frac{1}{6-2n} = \frac{1}{4}$, or the body goes off *ad inf.* after $\frac{1}{4}$ of a revolution. This must be considered as the limit, or the least angle at which the body can go off *ad infinitum;* for $n$ cannot be assumed accurately $=1$, since in this case the $f. \frac{-\dot{x}}{x^n}$ cannot be taken by the common method.

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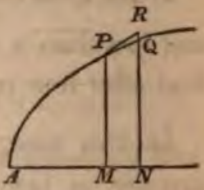
## SECTION V.

### (165.) PROPOSITION.

If a body describe a curve, when urged by a force acting in the direction of the ordinates, that force is proportional to the second fluxion of the ordinate.



Let $PR$ be a tangent to the curve at $P$; and $MP$, $NQR$, two ordinates indefinitely near to each other, in the direction of the force. The body at $P$ would describe $PR$, if the force did not act in the same time that it now describes $PQ$; therefore $RQ$ represents the small space through which it is impelled by the force in the time of describing $PQ$. Now, ultimately, when the time is very small, $S \propto F \times T^2$; therefore, if $T$ be assumed a small given time, $RQ \propto F$; but ultimately $RQ = \pm\frac{1}{2}\ddot{y}$, if $MP = y$ (Art. 97.); $\therefore F \propto \pm\ddot{y}$, according as the force is repulsive or attractive.

## EXAMPLES.

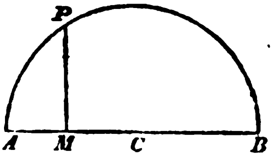
Ex. 1. A body describes a curve, whose equation is $x^5 = a^2y^3$, by a force acting in the direction of the ordinates. Required the law.

Here $y \propto x^{\frac{5}{3}}$; $\therefore \dot{y} \propto x^{\frac{2}{3}}\dot{x}$, $\propto x^{\frac{2}{3}}$, since the motion in a direction parallel to the axis must in these cases be uniform. Therefore $\ddot{y} \propto x^{\frac{1}{3}}$; that is, the force $\propto x^{\frac{1}{3}}$, or $\propto y^{\frac{1}{5}}$, and is repulsive; or the curve is convex to its axis.

Ex. 2. Let the equation to the curve be $y^{m+n} = a^m x^n$, and act in the direction of the ordinates. To find the law.

Here $y \propto x^{\frac{n}{m+n}}$; $\therefore \dot{y} \propto x^{\frac{-m}{m+n}}$, and $\ddot{y} \propto -x^{\frac{-2m+n}{m+n}}$, $\propto -\frac{1}{x^{\frac{2m+n}{m+n}}} \propto -\frac{1}{y^{\frac{2m+n}{n}}}$, and is attractive; or the curve is concave to the axis.

Ex. 3. A body describes a semi-circle by the action of a force in the direction of the ordinates. Required the law.



If $AM=x$, $AC=a$, $PM=y$,

$$y=\sqrt{2ax-x^2};$$

$$\therefore\ \dot{y}=\overline{a-x}\times\dot{x}\times\overline{2ax-x^2}|^{-\frac{1}{2}}$$

$$=\overline{a-x}\times\overline{2ax-x^2}|^{-\frac{1}{2}}$$

and $\ddot{y}=-\dot{x}\times\overline{2ax-x^2}|^{-\frac{1}{2}}-\frac{\overline{a-x}|^2}{\overline{2ax-x^2}|^{\frac{3}{2}}}\times\dot{x}=-\frac{\overline{2ax-x^2}+\overline{a-x}|^2}{\overline{2ax-x^2}|^{\frac{3}{2}}}$

$=\frac{-a^2}{y^3}$; therefore the force $\propto\frac{1}{PM^3}$. The same conclusion is deduced from the expression $\frac{\dot{p}}{p^3\dot{x}}$. Art. 153. Ex. 8.

Ex. 4. Let the curve be a parabola, and the force act as before.

In the parabola, $y=a^{\frac{1}{2}}x^{\frac{1}{2}}$, $\propto x^{\frac{1}{2}}$; $\therefore\ \dot{y}\propto x^{-\frac{1}{2}}$; $\therefore\ -\ddot{y}\propto x^{-\frac{3}{2}}$ $\propto\frac{1}{y^3}$; or the force $\propto\frac{1}{y^3}$.

Ex. 5. Let the curve be an ellipse.

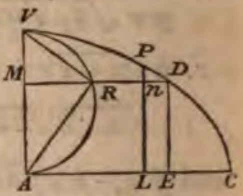
Here $y=\frac{b}{a}\times\sqrt{2ax-x^2}\propto\sqrt{2ax-x^2}$; hence, the reasoning and the conclusion is the same as in the circle, and the force $\propto\frac{1}{y^3}$.

Ex. 6. Let the curve be an hyperbola.

Here $y=\frac{b}{a}\times\sqrt{2ax+x^2}\propto\sqrt{2ax+x^2}$; and $\dot{y}\propto\overline{a+x}\times\overline{2ax+x^2}|^{-\frac{1}{2}}$; $\therefore\ \ddot{y}\propto\frac{2ax+x^2-\overline{a+x}|^2}{\overline{2ax+x^2}|^{\frac{3}{2}}}$; and $-\ddot{y}\propto\frac{1}{\overline{2ax+x^2}|^{\frac{3}{2}}}$ $\propto\frac{1}{y^3}$.

Ex. 7. Let the curve be a common cycloid *VDC*, and the force act in the direction of the ordinates, parallel to the axis.

Let $DE$, and $PL$, be two ordinates indefinitely near to each other; draw $DM$ parallel to $CA$, meeting the axis in $M$, and the generating circle in $R$; join $VR$, $AR$, and let $CE=x$, $ED=y$, $Dn=\dot{x}$, $Pn=\dot{y}$; and $AV=2a$. Then, by similar triangles, $PDn$, $RVM$, $\dot{x} : \dot{y} :: RM : MV :: AM : MR :: y : \sqrt{2ay-y^2}$; $\therefore\ \dot{y} = \dot{x} \times \dfrac{\sqrt{2ay-y^2}}{y}$; $\therefore\ \ddot{y} = \dfrac{\overline{a\dot{y}-y\dot{y}} \times y - \dot{y} \times \overline{2ay-y^2}}{y^2 \times \sqrt{2ay-y^2}}$ $= \dfrac{-ay\dot{y}}{y^2\sqrt{2ay-y^2}}$. But $\dot{x} = \dfrac{y\dot{y}}{\sqrt{2ay-y^2}}$; $\therefore\ \ddot{y} = \dfrac{-a\dot{x}}{y^2}$; and $-\ddot{y} \propto \dfrac{1}{y^2}$, or the force $\propto \dfrac{1}{y^2}$.

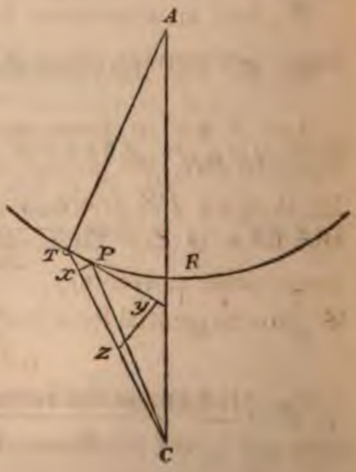


## SECTION VI.

(166.) PROPOSITION.

To find the forces, by which bodies may be made to perform isochronous oscillations in curve lines, when the force tends to a center.

Let $AT$ represent a pendulum, $C$ the center of force, and $R$ the lowest point in the curve. By Art. 157. Ex. 8. Cor. 3., if the force acting upon $T$ in a tangential direction be proportional to $TR$, the oscillations will be isochronous. Join $CT$; draw $Ty$ a tangent to the curve; take $Ty=TR$, and from $Y$ let $yz$ be drawn perpendicular to $Ty$. Then, if $Tz$ represent the force tending to $C$, the oscillations will be isochronous.



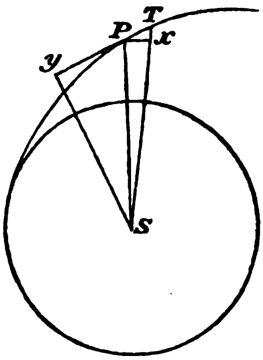
Take *CP* very near *CT*, and draw *Px* perpendicular to *CT*. Let $RT=z$, $CT=x$, $Tz=F$, $TP=\dot{z}$, $Tx=\dot{x}$. Then, by similar triangles, $F$ : the tangential force $Ty$ :: $\dot{z}$ : $\dot{x}$;

$$\therefore F=\frac{\text{the tangential force}\times\dot{z}}{\dot{x}}\propto\frac{z\dot{z}}{\dot{x}}.$$

### EXAMPLES.

Ex. 1. Let the force vary as the distance; then $z\dot{z}$ varies as $x\dot{x}$; $\therefore z^2$ varies as $x^2$, and $z$ varies as $x$, or the curve is the logarithmic spiral.

Ex. 2. Required the law of the force tending to *S*, which will make the involute of a circle the isochronous curve.

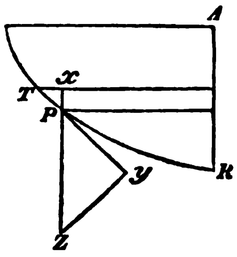


Here $\dot{z}:\dot{x}::ST:Ty::x:r$; $\therefore \frac{\dot{z}}{\dot{x}}=\frac{x}{r}$, and $\frac{z\dot{z}}{\dot{x}}=\frac{zx}{r}=F$; $\therefore F\times r=z\times x$; or, $F$ required : the arc :: the distance *ST* : the radius of the circle.

### (167.) PROPOSITION.

To find an expression for the force necessary to make any curve isochronous, when it acts in parallel lines.

Let it act in lines parallel to *AR*. Draw *Py* a tangent equal to *PR*, *yZ* perpendicular to *Py*, and let it meet *PZ* drawn in the direction of the force in *Z*. Then, making the same assumption as before,



$F$ : the tangential force :: $PZ$ : $Py$ :: $PT$ : $Px$ :: $\dot{z}$ : $\dot{x}$;

$$\therefore F=\frac{\text{the tangential force}\times\dot{z}}{\dot{x}}\propto\frac{z\dot{z}}{\dot{x}}.$$

### EXAMPLES.

Ex. 1. Let the force be constant. Then $\frac{z\dot{z}}{\dot{x}}$ varies as 1, and $x$ varies as $z^2$, a property of the common cycloid.

Ex. 2. Let $TR$ be a circular arc; to find the law. The force required : the tangential force :: $\dot{z} : \dot{x} :: r$ : sine; or, if the tangential force be represented by the arc, and gravity by the radius of the circle, $F$ : gravity :: arc : sine.

Ex. 3. Let the curve be the catenary, and the force act in a direction parallel to the axis.

In this curve, $z^2=2ax+x^2$; $\therefore z\dot{z}=a\dot{x}+x\dot{x}$. Hence $F$, or $\frac{z\dot{z}}{\dot{x}}=a+x$; therefore this force : gravity :: $a+x$ : $a$.

# CHAP. XXI.

## ON THE MOTION OF BODIES IN RESISTING MEDIUMS.

(168.) In estimating the effects of fluids upon the motion of bodies, the retardation may be conceived to vary according to any law of the velocity. Thus, in the second Book of the *Principia*, Sect. 1., it is supposed to be proportional to the velocity itself; and any other power might be assumed at pleasure. But it appears by experiment, that the resistance opposed to a given plane surface varies nearly as $V^2$; so that every other law of variation is to be considered rather as a mathematical hypothesis, than as founded upon fact. We are not sufficiently acquainted with the nature of fluids, to ascertain precisely in what manner they act. Hence, in demonstrating the theory, Sir I. NEWTON has introduced the following conditions. (Lib. II. Prop. 40., &c.)

1. That the particles of the fluid, in which the body moves, are perfectly non-elastic, and

2. That the fluid is infinitely compressed.

The first property belongs to mercury and water, and, provided the bodies move slowly, even to air; for the particles may then be considered as sliding away after impact, without adhesion to the body. If the particles were elastic, an additional resistance would be produced by the rebound.

The second supposition is not strictly true of any fluid. If a body move in a medium with considerable velocity, the parts left by it, as it advances, will not immediately be filled; and an additional resistance will arise from this cause. Thus, if its velocity in air be greater than that with which air rushes into an exhausted receiver, a vacuum will be left behind. Where the velocity is considerably greater, the air is condensed before the body, and, in proportion to the compression, exerts a force of elasticity against it. But in experiments where the velocity is slow, the second condition may be admitted without sensible error.

It is therefore assumed, that no resistance is opposed to the moving body, except that which arises from the inertia of the particles displaced; the effects of elasticity, tenacity, and friction, if any friction exist, in philosophical experiments on air, water, and mercury, being scarcely perceptible.

(169.) It is found by experiment, that the resistance opposed to a plane surface moving in a fluid, in a direction perpendicular to the plane, is equal to the weight of a column of the fluid, whose base is the area of the plane, and height the space through which a body must fall by gravity to acquire its velocity. Therefore, if $A =$ the area of the plane surface, and $z =$ the height from which a body must fall to acquire its velocity, the resistance $= A \times z$; the density of the medium being assumed $= 1$. The same reasoning is true for a cylinder moving in a fluid, in the direction of its axis; the curved surface, which is in the direction of its motion, being supposed to have no tendency to accelerate or retard the particles of the fluid. Hence, if $d =$ the diameter of its base, the resistance opposed to it $= \frac{zpd^2}{4}$.

(170.) Now in fluids indefinitely compressed, the resistance on this cylinder : that on a globe of the same diameter :: 2 : 1; therefore the resitance on a globe moving with the same velocity $= \frac{zpd^2}{8}$.

(171.) The same distinction is to be noticed between resisting and retarding forces, which exists between moving and accelerating forces. The resistance opposed to a body moving in a fluid is proportional to the quantity of motion destroyed in a given time; whereas, the retarding force is measured by the velocity destroyed in that time, or it is proportional to the resistance divided by the quantity of matter.

For the sake of perspicuity, this distinction is observed in the two following Sections.

(172.) Hence, if $n =$ the density of a globe, whose diameter is $d$, or $\frac{pd^3n}{6}$ its quantity of matter, and the specific gravity of the fluid be 1, the retarding force $= \frac{zpd^2}{8} \div \frac{pd^3n}{6} = \frac{3z}{4nd}$; or, in terms of the velocity, since $V^2 = 4mz$, the retarding force $= \frac{3v^2}{16mnd}$.

It was thought expedient to give examples of both methods of calculation in the following Propositions.

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## SECTION I.

### (173.) PROPOSITION.

The force of resistance, which is opposed to a sphere, moving in a fluid with any given velocity, is to the force which would destroy the sphere's whole motion, in the same time, in which it describes uniformly $\frac{8}{3}$ parts of its diameter, as the density of the fluid to the density of the sphere. (NEWTON, Lib. II. Prop. 38.)

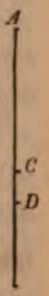
Let $M$ represent the resisting or moving force, which would destroy the sphere's motion in the specified time. Let $Q$ denote the quantity of matter in the sphere, $V$ its velocity, and $d$ the diameter; and suppose the specific gravity of the sphere is to that of the fluid as $n$ to 1. Then, since a body will describe $\frac{8d}{3}$ by an uniform velocity $V$, in the same time that it describes $\frac{4d}{3}$ when retarded by a constant force $\frac{M}{Q}$, which destroys that velocity, we have $V^2 = 4m \times \frac{M}{Q} \times \frac{4d}{3}$. But $V^2 = 4mz$, according to the notation of the preceding Chapter; therefore

$4m \times \frac{M}{Q} \times \frac{4d}{3} = 4mz$, and $M = \frac{3Qz}{4d}$. Now $Q = \frac{pd^3n}{6}$; therefore $M = \frac{zpd^2n}{8}$. But the resisting force opposed to the sphere $= \frac{zpd^2}{8}$ (Art. 170.); and this is to $\frac{zpd^2n}{8}$ :: 1 : $n$, or as the specific gravity of the fluid to that of the sphere.

### (174.) PROPOSITION.

Let a sphere of given diameter be projected in a resisting medium, whose specific gravity is to that of the sphere as 1 : $n$. Having given the velocity of projection at the point $A$, to find the velocity of the sphere at any given point $C$.

Let $AC = x$, $CD = \dot{x}$; then, since the retarding force $= \frac{3z}{4nd}$ (Art. 172.), $\dot{z} = -\frac{3z\dot{x}}{4nd}$; $\therefore \frac{\dot{z}}{z} = -\frac{3\dot{x}}{4nd}$; and the hyp. log. $z$ + corr. $= -\frac{3x}{4nd}$ + corr. When $x = 0$, let $z = a$; then the hyp. log. $\frac{z}{a} = -\frac{3x}{4nd}$. Hence, if $e = 2.71828$ &c., the number, whose hyperbolic logarithm is 1, we have $\frac{z}{a} = e^{\frac{-3x}{4nd}}$; $\therefore z = a \times e^{\frac{-3x}{4nd}}$; and the velocity at $C =$ $\sqrt{4mz} = \frac{\sqrt{4ma}}{e^{\frac{3x}{8nd}}}$.



COR. If the spaces be taken in arithmetic progression, the velocities are in inverse geometric.

### (175.) PROPOSITION.

To find the time of describing $AC$.

Here $\dot{T}$, which $= \frac{\dot{x}}{V} = \frac{\dot{x} \times e^{\frac{3x}{8nd}}}{\sqrt{4ma}}$, and $T = \frac{8nd}{3 \times \sqrt{4ma}}$ + corr. Let $T = 0$, then $x = 0$; $\therefore e^{\frac{3x}{8nd}} = e^0 = 1$. Hence, $T = \frac{8nd}{3 \times \sqrt{4ma}} \times$ $\overline{\times e^{\frac{3x}{8nd}} - 1}$.

### (176.) PROPOSITION.

If the retarding force, opposed to a sphere projected with a velocity $c$ in a resisting medium, vary as the velocity, the diminution of velocity is proportional to the space described. (NEWTON, Lib. II. Prop. 1.)

Let $r\left(=\frac{3c^2}{16mnd}\right.$, Art. 172.$\left.\right)$ represent the retarding force corresponding to the velocity $c$, and $v$ be any variable velocity; then, since $c : v :: r : \frac{rv}{c}$; $\therefore \frac{rv}{c}$ is the retarding force corresponding to the velocity $v$. Hence, $v\dot{v} = -2mF\dot{x} = -\frac{2mr}{c} \times v\dot{x}$; $\therefore \dot{x} = -\frac{c\dot{v}}{2mr}$, and $x +$ corr. $\propto$ corr. $-v$; but if $x=0$, $v=c$. Hence, $x \propto c-v$; or the space described varies as the velocity lost.

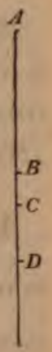
### (177.) PROPOSITION.

If the retarding force vary as in the last case, and times be taken in arithmetic progression, the velocities at the beginning of those times are in geometric progression. (NEWTON, Lib. II. Prop. 2.)

The same assumption being made, $\dot{x} = \frac{-c\dot{v}}{2mr}$; and $\dot{T} = \frac{\dot{x}}{V} = \frac{-c}{2mr} \times \frac{\dot{v}}{v}$; $\therefore T = -\frac{c}{2mr} \times$ hyp. log. $v +$ corr. Let $T=0$; then $v=c$. Hence, $T = \frac{c}{2mr} \times$ hyp. log. $\frac{c}{v}$; or $T \propto$ hyp. log. $\frac{c}{v}$.

If, therefore, the times be taken in arithmetic progression, the logarithms of the velocities are in arithmetic progression, and $\frac{c}{v}$, or $v$ itself, in geometric progression.

Cor. The spaces *AB*, *BC*, *CD*, &c. described in equal successive times, are in geometric progression. For, if *c*, *d*, *e*, *f*, &c. be the velocities at the beginnings of those times, they are by the Proposition in geometric progression. Hence also their differences, $c-d$, $d-e$, $e-f$, &c. are in geometric. But by the last Proposition, the spaces *AB*, *BC*, *CD*, &c. are proportional to these differences; therefore these spaces are in geometric progression.



The Corollary may be illustrated by the following process. Since $\dot{x} = -\frac{c\dot{v}}{2mr}$, $x = \frac{c}{2mr} \times \overline{c-v}$; that is, the first $x$ or *AB* $= \frac{c}{2mr} \times \overline{c-d}$. Again, since $x = \frac{c}{2mr} \times \overline{\text{corr.} - v}$, assume $x = AB$; then $v = d$; $\therefore\ x - AB$, or $BC = \frac{c}{2mr} \times \overline{d-e}$. In the same manner, $CD = \frac{c}{2mr} \times \overline{e-f}$; that is, the spaces *AB*, *BC*, *CD*, &c. are proportional to $c-d$, $d-e$, $e-f$, &c., or they are in geometric progression.

### (178.) Proposition.

If the retarding force vary as the $\overline{\text{vel.}}|^2$, and times be taken in geometric progression increasing, the velocities at the beginning of those several times are in the same geometric progression inverse; and the spaces described in these times are in arithmetic progression. (Newton, Lib. II. Prop. 5.)

Here $c^2 : v^2 :: r : \frac{rv^2}{c^2}$ = the retarding force corresponding to the velocity $v$. Hence, $v\dot{v} = -\frac{2mr}{c^2} \times v^2\dot{x}$; $\therefore\ \dot{x} = -\frac{c^2}{2mr} \times \frac{\dot{v}}{v}$, and $x = \frac{-c^2}{2mr} \times$ hyp. log. $v +$ corr. $= \frac{c^2}{2mr} \times$ hyp. log. $\frac{c}{v}$.

Also $\dot{T} = \frac{\dot{x}}{v} = -\frac{c^2}{2mr} \times \frac{\dot{v}}{v^2}$, and $T = \frac{c^2}{2mr} \times \frac{1}{v}$ + corr. = $\frac{c^2}{2mr} \times \overline{\frac{1}{v} - \frac{1}{c}}$. That is, if for $v$ we write $d$, $e$, $f$, &c., the time through the first space $AB$ (Fig. Art. 177.) $= \frac{c^2}{2mr} \times \overline{\frac{1}{d} - \frac{1}{c}}$; $T$, $BC = \frac{c^2}{2mr} \times \overline{\frac{1}{e} - \frac{1}{d}}$; $T$, $CD = \frac{c^2}{2mr} \times \overline{\frac{1}{f} - \frac{1}{e}}$, &c.; therefore, since the times are in geometric progression, $\frac{1}{d} - \frac{1}{c}$ and $\frac{1}{e} - \frac{1}{d}$, &c. are also; $\therefore \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c. are in geometric progression: that is, the velocities at the beginning of the several times are inversely in the same geometric progression as the times.

Also $x = \frac{c^2}{2mr} \times$ hyp. log. $\frac{c}{v}$; $\therefore$ the spaces in these times are in arithmetic progression.

This Proposition coincides in part with Art. 174.

### (179.) PROPOSITION.

Let a sphere be projected, as in the former cases, in a resisting medium, whose retarding force varies as the $\overline{\text{vel.}}|^n$; to find the velocity acquired through a given space, and the time of describing it.

Here $c^n : v^n :: r : \frac{rv^n}{c^n}$, the retarding force corresponding to the velocity $v$. Hence, $v\dot{v} = -\frac{2mr}{c^n} \times v^n \dot{x}$; let $\frac{1}{b} = \frac{2mr}{c^n}$, then $v\dot{v} = -\frac{v^n \dot{x}}{b}$; $\therefore \dot{x} = -bv^{1-n}\dot{v}$; and $x = -\frac{bv^{2-n}}{2-n}$ + corr.

Let $x = 0$; then $v = c$, and $x = \frac{b}{2-n} \times \overline{c^{2-n} - v^{2-n}}$; $\therefore v = \overline{\frac{bc^{2-n} - \overline{2-n}\,.\,x}{b}}\Big|^{\frac{1}{2-n}}$.

Also $\dot{T} = \frac{\dot{x}}{v} = -\frac{bv^{1-n}\dot{v}}{v} = -bv^{-n}\dot{v}$; $\therefore T = -\frac{bv^{1-n}}{1-n} + \text{corr.}$;

and the correct time $= \frac{b}{1-n} \times \overline{c^{1-n} - v^{1-n}}$.

Cor. 1. To find the space described before the velocity is wholly destroyed, assume $v=0$; then $x = \frac{b}{2-n} \times c^{2-n}$.

Cor. 2. The time, which elapses before the velocity is wholly destroyed $= \frac{b}{1-n} \times c^{1-n}$.

Cor. 3. If $n=2$, it coincides with Art. 178.

### (180.) Proposition.

If the retarding forces vary as $\overline{\text{vel.}}|^2$, and times be taken proportional to the first velocities directly, and the first retarding forces inversely, the velocities lost will be proportional to the whole, and the spaces described will be proportional to the times and the first velocities jointly. (Newton, Lib. II. Prop. 7.)

The retarding force corresponding to the velocity $v = \frac{rv^2}{c^2}$; $\therefore v\dot{v} = -\frac{rv^2\dot{x}}{c^2}$; and $\dot{x} = -\frac{c^2\dot{v}}{rv}$. Hence, $\dot{T} = \frac{\dot{x}}{v} = -\frac{c^2\dot{v}}{rv^2}$, and $T = \frac{c^2}{rv} + \text{corr.} = \frac{c^2}{rv} - \frac{c}{r} = \frac{c}{r} \times \overline{\frac{c-v}{v}}$. But by the Proposition, $T \propto \frac{c}{r}$; $\therefore v \propto c - v$, and $c - v \propto c$, or the velocity lost is proportional to the whole; and it is evident also that $c \propto v$.

Again, $\dot{T} = \frac{\dot{x}}{v}$; $\therefore \dot{x} = v \times \dot{T} \propto c \times \dot{T}$; $\therefore x \propto c \times T$.

## SECTION II.

(181.) In the last Section, the bodies were supposed to be influenced by no forces except those of projection and resistance; in which case the velocity is continually retarded. In this Section they are supposed to be acted upon by gravity, or by some force tending to a center. This combined with the resistance may produce either an accelerated, or a retarded, or even a uniform velocity. The forces are supposed to act in the direction of the body's first motion, and the motion is consequently rectilinear.

### (182.) PROPOSITION.

Let a spherical body descend in a fluid from rest by the action of gravity, and let the specific gravity of the sphere be to that of the fluid as $n : 1$; to find the greatest velocity which the sphere can acquire.

The same assumption being made, as in Art. 172., the difference of the weights of the sphere and an equal bulk of the fluid, or the absolute force of the body's descent $= \frac{pd^3n}{6} - \frac{pd^3}{6}$; this is the same, whatever be the velocity of the globe, the fluid being infinitely compressed. Now, let $z$ be the space due to the velocity at any point of the descent; the resistance opposed on this account $= \frac{zpd^2}{8}$ (Art. 170.); $\therefore$ the whole resistance $= \frac{pd^3n}{6} - \frac{pd^3}{6} - \frac{zpd^2}{8}$. Divide this by $\frac{pd^3n}{6}$ the mass moved, and the accelerating force $= 1 - \frac{1}{n} - \frac{3z}{4nd}$; but when the velocity is a maximum, the accelerating force $= 0$; $\therefore 1 - \frac{1}{n} - \frac{3z}{4nd} = 0$. Hence, $z = \frac{4d \times \overline{n-1}}{3}$; and $v = \sqrt{4mz} = \sqrt{\frac{16md \times \overline{n-1}}{3}}$.

### (183.) PROPOSITION.

The greatest velocity which can be acquired by a spherical body descending in a fluid, is equal to that which the body would acquire in vacuo by its comparative gravity in falling through a space, which is to $\frac{4d}{3}$ :: the density of the sphere : the density of the fluid. (NEWTON, Lib. II. Prop. 38. Cor. 2.) For take $x : \frac{4d}{3} :: n : 1$; then, $x = \frac{4nd}{3}$. Also $\frac{pd^3n}{6} - \frac{pd^3}{6}$, divided by $\frac{pd^3n}{6}$, or $1 - \frac{1}{n}$, or $\frac{n-1}{n}$ = the force of the sphere's acceleration in the fluid, or the force of its comparative gravity. And the velocity acquired through $\frac{4nd}{3}$ by the action of this force in vacuo $= \sqrt{\frac{16mnd}{3} \times \frac{n-1}{n}} = \sqrt{\frac{16md \times \overline{n-1}}{3}}$.

### (184.) PROPOSITION.

Let a spherical body descend in a fluid from rest, and let the specific gravity of the body be to that of the fluid as $n$ to 1; to find the velocity at any point of the descent.

By Art. 182, the accelerating force $= 1 - \frac{1}{n} - \frac{3z}{4nd}$; hence, if $x$ = the space described, $\dot{z} = \dot{x} - \frac{\dot{x}}{n} - \frac{3z\dot{x}}{4nd}$; $\therefore \frac{\dot{x}}{4nd} = \frac{\dot{z}}{4nd - 4d - 3z} = \frac{\dot{z}}{4d \times \overline{n-1} - 3z}$, and $\frac{x}{4nd} = -\frac{1}{3} \times$ hyp. log. $\overline{4d.\overline{n-1} - 3z}$ + corr. Now let $x = 0$, then $z = 0$; $\therefore c = \frac{1}{3} \times$ hyp. log. $4d.\overline{n-1}$; consequently $\frac{x}{4nd} = -\frac{1}{3} \times$ hyp. log. $\frac{4d.\overline{n-1} - 3z}{4d.\overline{n-1}}$.. Let $e$ = the number whose hyp. log. is 1; then $e^{\frac{-3x}{4nd}} = \frac{4d.\overline{n-1} - 3z}{4d.\overline{n-1}} = 1 - \frac{3z}{4d.\overline{n-1}}$; $\therefore z = \frac{4d}{3} \times$

$\overline{n-1}.\overline{1-e^{\frac{-3x}{4nd}}}$; and the velocity, or $\sqrt{4mz} = \sqrt{\frac{16md.\overline{n-1}}{3}}$ $\times \sqrt{1-e^{\frac{-3x}{4nd}}}$.

Cor. If $x$ be increased *sine limite*, $e^{\frac{-3x}{4nd}}$ will be a quantity indefinitely small; and the ultimate velocity of the sphere will $= \sqrt{\frac{16md \times \overline{n-1}}{3}}$, as in Art. 182.

### (185.) Proposition.

If a spherical body of given diameter be immersed in a fluid, and its specific gravity be indefinitely less than that of the fluid, the velocity of ascent will be uniform, and equal to that which a heavy body would acquire in falling from rest through $\frac{4}{3}$ of the diameter by the action of gravity.

Suppose the sphere to have ascended from rest through a space $=x$, and let $z$ be the space due to the velocity at that point; then, if $n =$ the specific gravity of the sphere, its weight $= \frac{pd^3n}{6}$, and the resistance to its motion arising from the velocity $= \frac{zpd^2}{8}$; $\therefore$ the whole resistance opposed to the sphere's ascent $= \frac{pd^3n}{6} + \frac{zpd^2}{8}$. But its force of ascent is the weight of a quantity of fluid equal in magnitude to the sphere, and this weight $= \frac{pd^3}{6}$, the specific gravity being 1. Hence the force of the sphere's ascent $= \frac{pd^3}{6} - \frac{pd^3n}{6} - \frac{zpd^2}{8}$; divide this by the sphere's weight, and we get the accelerating force $= \frac{1}{n} - 1 - \frac{3z}{4nd} = \frac{1}{n} - \frac{3z}{4nd}$, for 1 vanishes in respect of the other terms. Hence, $\dot{z} = \frac{\dot{x}}{n} - \frac{3z\dot{x}}{4nd}$, and $\frac{\dot{x}}{4nd} = \frac{\dot{z}}{4d-3z}$; $\therefore \frac{x}{4nd} = -\frac{1}{3} \times$ hyp. log. of $\overline{4d-3z}$ + corr. Let $x = 0$, then $z=0$; $\therefore$ corr. $= \frac{1}{3}$ hyp. log. $4d$; $\therefore \frac{x}{4nd} = -\frac{1}{3} \times$ hyp. log. $\frac{4d-3z}{4d}$.

Hence, assuming $e$ as before, the number whose hyperbolic logarithm is 1, we have $\frac{4d - 3z}{4d} = e^{\frac{-3x}{4nd}} = 0$, for $\frac{3x}{4nd}$ is indefinitely great; $\therefore z = \frac{4d}{3}$; and the velocity at this point $= \sqrt{4mz} = \sqrt{\frac{16md}{3}}$, which is the same with that acquired by gravity through $\frac{4d}{3}$.

As this reasoning is equally true for all points of the ascent, the velocity of the body is uniform.

Cor. Spherical bodies without weight, of different diameters, ascend in fluids with uniform velocities, which vary as the square root of the diameters.

### (186.) Proposition.

Let a body descend or be projected in a resisting medium directly toward a center of force, and be attracted by a constant force toward the center; to find the greatest velocity which it can acquire, the retarding force of the medium being supposed to vary as the velocity.

Let $F$ represent the constant force; $r =$ the retarding force corresponding to the velocity $c$, and $v$ any variable velocity. Then the retarding force corresponding to $v = \frac{rv}{c}$; $\therefore$ the whole force of acceleration toward the center $= F - \frac{rv}{c}$. But when the velocity is a maximum, the force of acceleration $= 0$; $\therefore F - \frac{rv}{c} = 0$, and $v = \frac{cF}{r}$.

Cor. The greatest velocity $v$ : the velocity $c$ :: $F$ : the resistance $r$ corresponding to $c$. (Newton, Lib. II. Prop. 3. Cor. 1.)

### (187.) Proposition.

Let a body descend or be projected in a resisting medium under the same circumstances as in the last case, toward the center of force. To find the space and time corresponding to any velocity. (Newton, Lib. II. Prop. 3.)

The accelerating force upon the descending body, by the last Proposition, $= F - \frac{rv}{c} = F - av$, where $a = \frac{r}{c}$. Hence, if $x$ be the space described, $v\dot{v} = 2m\dot{x} \times \overline{F - av}$; $\therefore\ \dot{x} = \frac{v\dot{v}}{2m \times \overline{F - av}}$ $= \frac{v\dot{v}}{2ma \times \overline{\frac{F}{a} - v}}$. Let $\frac{F}{a} = b$; then $\dot{x} = \frac{v\dot{v}}{2ma \times \overline{b - v}} = \frac{1}{2ma}$ $\times : \dot{v} + \frac{v\dot{v}}{b - v} - \dot{v} = \frac{1}{2ma} \times : \frac{b\dot{v}}{b - v} - \dot{v}$, and $x = \frac{1}{2ma} \times :$ $-b \times$ hyp. log. $\overline{b - v} - v +$ corr. Let $x = 0$; then, if the body descend from rest, $v = 0$; and $x$ corrected $= \frac{b}{2ma} \times :$ hyp. log. $\frac{b}{b - v} - \frac{1}{2ma} \times v$. But if it be projected, let the velocity of projection $= c$; then $x = \frac{b}{2ma} \times$ hyp. log. $\frac{b - c}{b - v}$ $+ \frac{1}{2ma} \times \overline{c - v}$.

Also, $\dot{T} = \frac{\dot{x}}{v} = \frac{\dot{v}}{2ma \times \overline{b - v}}$; $\therefore\ T = -\frac{1}{2ma} \times$ hyp. log. $\overline{b - v}$ + corr.; therefore, if the body descend from rest, the correct time $= \frac{1}{2ma} \times$ hyp. log. $\frac{b}{b - v}$. If it be projected with the velocity $c$, $T = \frac{1}{2ma} \times$ hyp. log. $\frac{b - c}{b - v}$.

### (188.) PROPOSITION.

The same supposition remaining, let the body be projected directly from the center of force. To find the space and time corresponding to any velocity.

In this case, the force of retardation $= F + \frac{rv}{c}$, and $v\dot{v} =$ $-2m\dot{x} \times \overline{F + \frac{rv}{c}} = -2m\dot{x} \times \overline{F + av}$, and $\dot{x} = \frac{1}{2m} \times \frac{-v\dot{v}}{F + av}$ $= \frac{1}{2ma} \times \frac{-v\dot{v}}{b + v}$; and, as in the last Proposition, $x = \frac{b}{2ma}$ $\times$ hyp. log. $\frac{b + v}{b + c} + \frac{1}{2ma} \times$ ~~$\overline{v - c}$.~~ $\overline{c - v}$.

Also $\dot{T} = \frac{\dot{x}}{v} = -\frac{1}{2ma} \times \frac{\dot{v}}{b+v}$; $\therefore T = -\frac{1}{2ma} \times$ hyp. log. $\overline{b+v} + \frac{1}{2ma} \times$ hyp. log. $\overline{b+c} = \frac{1}{2ma} \times$ hyp. log. $\frac{b+c}{b+v}$.

### (189.) PROPOSITION.

Let a body descend or be projected toward a center as before, the retarding force of the medium varying as the square of the velocity. To find the greatest velocity which the body can acquire.

The retarding force corresponding to the velocity $v$ is in this case $\frac{rv^2}{c^2}$; $\therefore$ the whole force of acceleration $= F - \frac{rv^2}{c^2}$. But when the velocity is a maximum, this force $= 0$; $\therefore v = c \times \sqrt{\frac{F}{r}}$.

COR. 1. $V : c :: \sqrt{F} : \sqrt{r}$. (NEWTON, Lib. II. Prop. 8. Cor. 3.)

COR. 2. Since $r = \frac{3c^2}{16mnd}$ (Art. 172.), if $F$ represent the comparative gravity of the body, or $= \frac{n-1}{n}$, $v = c \times \sqrt{\frac{n-1}{n} \times \frac{16mnd}{3c^2}} = \sqrt{16md \times \frac{\overline{n-1}}{3}}$; and this Proposition coincides with Art. 182.

### (190.) PROPOSITION.

Let a body be projected in a resisting medium directly towards a center of force under the same supposition as in the last case. To find the space and time corresponding to any velocity.

The accelerating force in this case corresponding to the velocity $v = F - \frac{rv^2}{c^2} = F - av^2$, if $a = \frac{r}{c^2}$; $\therefore v\dot{v} = 2m\dot{x} \times$

$\overline{F - av^2}$; and $\dot{x} = \dfrac{v\dot{v}}{2m \times \overline{F - av^2}} = \dfrac{1}{2ma} \times \dfrac{v\dot{v}}{b^2 - v^2}$, where $b^2 = \dfrac{F}{a}$; $\therefore x = -\dfrac{1}{4ma} \times$ hyp. log. $b^2 - v^2 +$ corr. When $x = 0$, let $v = c$; $\therefore x = \dfrac{1}{4ma} \times$ hyp. log. $\dfrac{b^2 - c^2}{b^2 - v^2}$.

Also $\dot{T} = \dfrac{\dot{x}}{v} = \dfrac{\dot{v}}{2ma \times \overline{b^2 - v^2}}$; $\therefore T = \dfrac{1}{4mab} \times$ hyp. log. $\dfrac{b + v}{b - v} +$ corr. When $T = 0$, let $v = c$; $\therefore T = \dfrac{1}{4mab} \times$ : hyp. log. $\dfrac{b + v}{b - v}$ − hyp. log. $\dfrac{b + c}{b - c}$.

If the body descend from rest, in correcting the fluents, $v$ must be assumed $= 0$.

(191.) The first part of this Proposition will coincide with Art. 184, by taking the fluent without correction, and substituting the proper values for $a$ and $b$, and $F$ and $r$.

Thus, $\dot{x} = \dfrac{1}{2ma} \times \dfrac{v\dot{v}}{b^2 - v^2}$; $\therefore x = -\dfrac{1}{4ma} \times$ hyp. log. $b^2 - v^2 +$ corr. Let $x = 0$, and $v = 0$, or the body descend by the force $F$, its comparative gravity; then, $x = -\dfrac{1}{4ma} \times$ hyp. log. $\dfrac{b^2 - v^2}{b^2}$. Hence, if $e$ be the number whose hyperbolic logarithm is 1, we have $e^{-4max} = \dfrac{b^2 - v^2}{b^2}$.

Now for $a$ write $\dfrac{r}{c^2}$, or $\dfrac{3c^2}{16mnd} \times \dfrac{1}{c^2}$ (Art. 172.), or $\dfrac{3}{16mnd}$; for $F$ write $\dfrac{n - 1}{n}$, and for $b^2$ the fraction $\dfrac{F}{a}$; then $v^2$ will $= 16md \times \dfrac{\overline{n-1}}{3} \times \overline{1 - e^{\frac{-3x}{4nd}}}$; and $v = \sqrt{\dfrac{16md.\overline{n-1}}{3}} \times \sqrt{1 - e^{\frac{-3x}{4nd}}}$.

### (192.) PROPOSITION.

The same supposition remaining, let the body be projected directly from the center of force. To find the space and time corresponding to any velocity.

The retarding force in this case $= F + \frac{rv^2}{c^2}$, and $v\dot{v} = -2m\dot{x} \times \overline{F + \frac{rv^2}{c^2}} = -2m\dot{x} \times \overline{F + av^2}$; $\therefore\ \dot{x} = -\frac{1}{2m} \times \frac{v\dot{v}}{F + av^2} = -\frac{1}{2ma} \times \frac{v\dot{v}}{b^2 + v^2}$; and $x = -\frac{1}{4ma} \times$ hyp. log. $b^2 + v^2$ + corr. $= \frac{1}{4ma} \times$ hyp. log. $\frac{b^2 + c^2}{b^2 + v^2}$, where $c$ is the velocity of projection.

Also, $\dot{T} = \frac{\dot{x}}{v} = -\frac{1}{2ma} \times \frac{\dot{v}}{b^2 + v^2}$; $\therefore\ T = -\frac{1}{2mab^2} \times$ a circular arc, whose radius is $b$, and tangent $v +$ corr. Let this arc $= N$. Now if $T = 0$, $v = c$. Let the arc, whose radius is $b$, and tangent $c = M$; then $T$ corrected $= \frac{1}{2mab^2} \times \overline{M - N}$.

Cor. 1. To find the greatest height to which the body will rise, take $v = 0$; then $x = \frac{1}{4ma} \times$ hyp. log. $\frac{b^2 + c^2}{b^2} =$ by substituting the values of $a$ and $b^2$, as in the Proposition, $\frac{c^2}{4mr} \times$ hyp. log. $\frac{F + r}{F}$.

Cor. 2. To find the time, in which the body will lose all its velocity, assume $v = 0$; then the arc $N = 0$; $\therefore\ T = \frac{1}{2mab^2} \times M$.

Cor. 3. Since the time in which the body loses its whole velocity varies as $M$, it varies as $b \times M$; or as the sector of a circle, whose radius is $b$, and tangent $c$. (Newton, Lib. II. Prop. 9.)

Cor. 4. And the time in which the velocity is diminished by the quantity $c - v$ varies as $M - N$, or as $b \times \overline{M - N}$; that is, it is proportional to the difference of two sectors, whose tangents are $c$ and $v$, to the same given radius $b$.

### (193.) Proposition.

The same supposition being made, if spaces be taken in arithmetic progression, the whole force which accelerates or retards, according as the body descends or ascends, will be in geometric progression. (Newton, Lib. II. Prop. 8.)

1. By Art. 190, if the body be projected downward with a velocity $c$, $x = \frac{1}{4ma} \times$ hyp. log. $\frac{b^2-c^2}{b^2-v^2}$; if it descend from rest, $x = \frac{1}{4ma} \times$ hyp. log. $\frac{b^2}{b^2-v^2}$. In either case, if $x$ be in arithmetic progression, the fractional part of the expression is in geometric. Hence, since the numerator is constant, $b^2-v^2$, or $\frac{F}{a}-v^2$, or $F-av^2$, or its equal $F-\frac{rv^2}{c^2}$; the whole force of acceleration is also in geometric progression.

2. Let the body be projected from the center, or ascend. Here (by Art. 192.) $x = \frac{1}{4ma} \times$ hyp. log. of $\frac{b^2+c^2}{b^2+v^2}$; therefore, if $x$ be in arithmetic progression, $b^2+v^2$, or $\frac{F}{a}+v^2$, or $F+\frac{rv^2}{c^2}$, the whole of the retarding force is in geometric progression.

### (194.) Proposition.

In general, let a body be projected with a given velocity in a resisting medium directly to, or directly from, a center of force; and let it be attracted by a constant force to that center. To find the space and time corresponding to any velocity, the retarding force being supposed to vary as the $\overline{\text{vel.}}|^{n}$.

1. If the body descend, the whole force of acceleration $= F-\frac{rv^n}{c^n}$; $\therefore\ v\dot{v} = 2m\dot{x} \times \overline{F-\frac{rv^n}{c^n}}$, and $\dot{x} = \frac{v\dot{v}}{2m \times \overline{F-\frac{rv^n}{c^n}}}$;

Also, $\dot{T} = \frac{\dot{x}}{v} = \frac{\dot{v}}{2m \times \overline{F-\frac{rv^n}{c^n}}}$.

2. If the body ascend, the whole force of retardation $= F+\frac{rv^n}{c^n}$; and $v\dot{v} = -2m\dot{x} \times \overline{F+\frac{rv^n}{c^n}}$, and $\dot{x} = \frac{-v\dot{v}}{2m \times \overline{F+\frac{rv^n}{c^n}}}$.

Also, $\dot{T} = \frac{\dot{x}}{v} = \frac{-\dot{v}}{2m \times \overline{F + \frac{rv^n}{c^n}}}$.

The fluents must be found for the particular cases.

(195.) These expressions may be applied to the descent of bodies in resisting mediums at the surface of the earth, by substituting for $F$ the fraction $\frac{n-1}{n}$, which represents the force of its comparative gravity. An instance of this is given in Art. 189.

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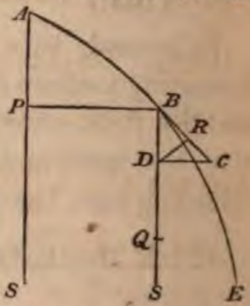
## SECTION III.

(196.) PROPOSITION.

To determine the resistance of a medium, by which a body when impelled by a given force $F$, acting in parallel lines, may describe a given curve.

Let $ABE$ be the given curve; $BS$ the direction of the force at $B$. Draw $BP$ perpendicular to $BS$, meeting the axis $AS$, or some given line parallel to $BS$, in $P$. Take $BQ = s = \frac{1}{2}$ the chord of curvature at $B$. Let $AP = x$, $BD = \dot{x}$, $AB = z$, $BC = \dot{z}$; and draw $DR$ perpendicular to $BC$.



Now a body must fall through $\frac{1}{4}$th of the chord of curvature by the constant force $F$, to acquire the velocity in the curve, whether it moves in vacuo, or in a resisting medium. For the resistance has no effect in causing the body to deviate from the tangent; it only retards its motion in such a manner, that it may always bear a just proportion to the given force $F$. Hence, $v = \sqrt{2mFs}$, and $\dot{v} = \frac{mF\dot{s}}{\sqrt{2mFs}}$ = the whole increase of velocity in the direction $BC$, in the time of describing $BC$, or in a time $= \frac{\dot{z}}{v}$. Also, the increase in the direction $BS$ by the force $F$ in that time $= 2mF\dot{t} = \frac{2mF\dot{z}}{\sqrt{2mFs}}$; and this increase : the increase of velocity in the direction $BC$ from that cause :: $BD : BR :: BC : BD :: \dot{z} : \dot{x}$; $\therefore$ the increase in the direction $BC = \frac{2mF\dot{z}}{\sqrt{2mFs}} \times \frac{\dot{x}}{\dot{z}} = \frac{2mF\dot{x}}{\sqrt{2mFs}}$; hence, the effect of the resistance $= m \times \frac{F\dot{s} - 2F\dot{x}}{\sqrt{2mFs}}$, the difference of these increments; or the decrement of velocity arising from the resistance $= -m \times \frac{F\dot{s} - 2F\dot{x}}{\sqrt{2mFs}}$. Therefore, since $F \propto$ the increment or decrement of velocity generated in a given time, the resistance : the force $F$ :: $-m \times \frac{F\dot{s} - 2F\dot{x}}{\sqrt{2mFs}} : \frac{2mF\dot{z}}{\sqrt{2mFs}}$ $:: \frac{\dot{s} - 2\dot{x}}{2z} : 1$, by omitting the negative sign, which denotes a retarding force.

(197.) To obtain an expression in terms of the abscissa and the curve. Let $y = PB$; then $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, and $s = \frac{\dot{z}^2}{\ddot{x}}$ (Art. 101.) $= \frac{\dot{x}^2 + \dot{y}^2}{\ddot{x}}$; therefore, if $\dot{y}$ be assumed constant,
$\dot{s} = \frac{2\dot{x}\ddot{x}^2 - \overline{\dot{x}^2 + \dot{y}^2} \times \dot{\ddot{x}}}{\ddot{x}^2} = \frac{2\dot{x}\ddot{x}^2 - \dot{z}^2\dot{\ddot{x}}}{\ddot{x}^2}$; hence, $\frac{\dot{s} - 2\dot{x}}{2\dot{z}} = -\frac{\dot{z}\dot{\ddot{x}}}{2\ddot{x}^2}$. Therefore, the resistance : the force $F$ :: $\frac{\dot{z}\dot{\ddot{x}}}{2\ddot{x}^2} : 1$, the negative sign being omitted as before.

### (198.) PROPOSITION.

To determine the resistance by which a body may describe any given curve about a center of force.

In this case, $F$ is considered as variable. Hence, by making the same supposition as before, since $v = \sqrt{2mFs}$, $\dot{v} = m \times$

$\frac{F\dot{s}+s\dot{F}}{\sqrt{2mFs}}$ = the whole increase of velocity in a tangential direction, during the time $\frac{\dot{z}}{\sqrt{2mFs}}$. Also, the increase in the same direction by the force $F$ in the same time $= \frac{2mF\dot{x}}{\sqrt{2mFs}}$, as before; or if $SB = w$, since $\dot{x} = -\dot{w}$, this increase $= -\frac{2mF\dot{w}}{\sqrt{2mFs}}$; therefore the decrement of velocity, arising from the resistance, $= -m \times \frac{F\dot{s}+s\dot{F}+2F\dot{w}}{\sqrt{2mFs}}$; and this resistance : the force $F :: m \times \frac{F\dot{s}+s\dot{F}+2F\dot{w}}{\sqrt{2mFs}} : \frac{2mF\dot{z}}{\sqrt{2mFs}} :: \frac{F\dot{s}+s\dot{F}+2F\dot{w}}{2F\dot{z}} : 1$, the negative sign being again omitted.

Cor If the force act in parallel lines, or $S$ be at an infinite distance, $F$ is constant, and $\dot{F} = 0$; therefore, in this case, the resistance : $F :: \frac{F\dot{s}+2F\dot{w}}{2F\dot{z}} : 1 :: \frac{\dot{s}+2\dot{w}}{2\dot{z}} : 1 :: \frac{\dot{s}-2\dot{x}}{2\dot{z}} : 1$, the same proportion which was found in Art. 196.

### (199.) Examples.

Ex. 1. Let the curve be the common parabola, and the force act in lines parallel to $AS$.

Here $ax = y^2$; $\therefore a\dot{x} = 2y\dot{y}$; and $a\ddot{x} = 2\dot{y}^2 = 2$, if $\dot{y}$ be assumed constant, and $= 1$. Hence, $\dddot{x} = 0$; also, $\dot{z} = \frac{\sqrt{4y^2+a^2}}{a}$ (Art. 55.) $= \sqrt{\frac{4x+a}{a}}$; $\therefore \frac{\dot{z}\dddot{x}}{2\ddot{x}^2} = \frac{\sqrt{\frac{4x+a}{a}} \times 0}{2\ddot{x}^2}$, or the resistance is nothing.

Ex. 2. Let the curve be any parabola, whose equation is $a^{n-1}x = y^n$, the force acting in parallel lines as before.

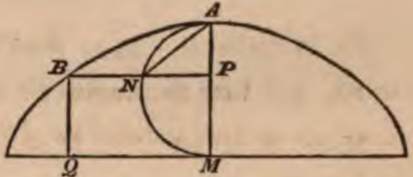
Here $\dot{x} = \frac{ny^{n-1}\dot{y}}{a^{n-1}}$; therefore, $\ddot{x} = \frac{n.\overline{n-1}.y^{n-2}\dot{y}^2}{a^{n-1}}$; and $\dddot{x} = \frac{n.\overline{n-1}.\overline{n-2}.y^{n-3}\dot{y}^3}{a^{n-1}}$. Also, $\dot{z} = \frac{\overline{n^2y^{2n-2}+a^{2n-2}}|^{\frac{1}{2}} \times \dot{y}}{a^{n-1}}$; therefore if $\dot{y} = 1$, the resistance, or $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$, $= \frac{n-2}{2n.\overline{n-1}} \times \frac{\overline{n^2y^{2n-2}+a^{2n-2}}|^{\frac{1}{2}}}{y^{n-1}}$, $F$ being assumed $=1$.

Cor. 1. If $n=2$, the resistance $=0$.

Cor. 2. If $n$ be greater than 1, but less than 2, the expression becomes negative; or the medium propels the body, instead of retarding it.

Cor. 3. If $n = 1$, the expression becomes negative, and infinite; that is, the propelling force of the medium is infinite, and the body moves in a right line.

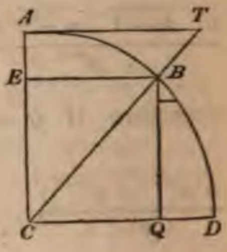
Ex. 3. Let the curve be a cycloid, and the force act in lines parallel to the axis $AM$.



Let $AM = a$. The perpendicular $BQ = \frac{1}{2}s$; therefore $\dot{x} = -\frac{1}{2}\dot{s}$, and $\dot{s} = -2\dot{x}$.

Also, $\dot{z} : \dot{x} :: AN : AP :: AM : AN :: \sqrt{AM} : \sqrt{AP} :: a^{\frac{1}{2}} : x^{\frac{1}{2}}$; $\therefore \dot{z} = \frac{a^{\frac{1}{2}}\dot{x}}{x^{\frac{1}{2}}}$; hence the resistance, or $\frac{\dot{s}-2\dot{x}}{2\dot{z}}$, $= -\frac{4\dot{x}}{2\dot{z}} = -\frac{4\dot{x}}{2} \times \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}\dot{x}} = -\frac{2x^{\frac{1}{2}}}{a^{\frac{1}{2}}} = -2 \times \sqrt{\frac{AP}{AM}} = -\frac{2AN}{AM} = \frac{2AN}{AM}$, the negative sign being omitted, and the force of gravity being assumed $=1$. The velocity $\propto \sqrt{BQ}$.

Ex. 4. Let the curve be a quadrant of a circle, the force as before.



Take $AE=x$, $AC=a$, $BQ=s$; then $s=a-x$; $\therefore\ \dot{s}=-\dot{x}$, and $\frac{\dot{s}-2\dot{x}}{2\dot{z}}=-\frac{3\dot{x}}{2\dot{z}}$. But $\dot{x} : \dot{z} :: BE : BC$; $\therefore -\frac{3\dot{x}}{2\dot{z}}=-\frac{3BE}{2BC}$, or changing the sign, the resistance $=\frac{3BE}{2BC}$, gravity being assumed $=1$.

Cor. 1. At $A$, $BE$ vanishes; therefore the resistance $=0$. At $C$, $BE=BC$; and the resistance : gravity :: 3 : 2. When $3BE=2BC$, the resistance $=1$, $=$ gravity.

Cor. 2. The velocity at $B \propto \sqrt{BQ}$; also the resistance varies as $BE$. Hence, if the resistance be supposed to vary as $V^2 \times$ the density of the medium, $BE \propto BQ \times$ the density; therefore, the density at $B \propto \frac{BE}{BQ} \propto \frac{BE}{CE} \propto \frac{AT}{AC} \propto$ the tangent of the arc $AB$. (Newton, Lib. II. Prop. 10.)

Ex. 5. Let the force tend to the center of a logarithmic spiral, and vary as the $\overline{\text{dist.}}|^{n}$.

Here $F \propto w^n$; $\therefore\ \dot{F} \propto nw^{n-1}\dot{w}$; and the expression $\frac{F\dot{s}+s\dot{F}+2F\dot{w}}{2F\dot{z}}$ for the resistance, since $s=w$, becomes $\frac{w^n\dot{w}+nw^n\dot{w}+2w^n\dot{w}}{2w^n\dot{z}}=\frac{n+3}{2}\times\frac{\dot{w}}{\dot{z}}$. But $\dot{w} : \dot{z}$ in a constant ratio, :: $a : b$; $\therefore\ \frac{\dot{w}}{\dot{z}}=\frac{a}{b}$; and the expression for the resistance is $\frac{n+3}{2}\times\frac{a}{b}$, the force tending to $S$ being assumed $=1$.

Cor. 1. If $n=-3$, or the force varies inversely as the cube of the distance, the resistance $=0$.

Cor. 2. If $n$ be a greater negative number than 3, the medium propels the body.

Cor. 3. If for $F$ we substitute $w^n$, then the resistance : the force $w^n$ :: $\frac{n+3}{2} \times \frac{d}{b}$ : 1; therefore the resistance $= \frac{n+3}{2} \times \frac{a}{b} \times w^n$.

Cor. 4. Since the velocity $\propto \sqrt{Fs}$, or as $w^{\frac{n+1}{2}}$, and the density of the medium varies as the resistance divided by the square of the velocity; the density in this case $\propto \frac{w^n}{w^{n+1}} \propto \frac{1}{w}$; therefore, conversely, if the density of the medium $\propto \frac{1}{w}$, the body may describe the logarithmic spiral, whatever be the value of $n$. (Newton, Lib. II. Prop. 16.)

Cor. 5. If $n = -2$, then the force varies as $\frac{1}{w^2}$; or $F$ varies as the square of the density. (Newton, Lib. II. Prop. 15.)

# CHAP. XXII.

## FLUENTS.

### SECTION I.

(200.) METHODS have been already proposed for finding, in certain cases, the fluent from the fluxion. The following Chapter is intended to furnish a variety of Examples, with Rules to facilitate the operation.

RULE. The fluents of such fluxions as are of the form $\frac{x^m\dot{x}}{a\pm x}$, $\frac{x^m\dot{x}}{a^2\pm x^2}$, where $m$ is a whole positive number; and those of the form $\frac{x^{rn-1}\dot{x}}{a^n\pm x^n}$, where $r$ is a whole positive number, may be found by dividing the numerator by the denominator in an inverted order.

(201.) **FLUENT 1.**

1. To find the fluent of $\frac{x\dot{x}}{a+x}$.

$$\begin{array}{r|l} x+a) & x\dot{x}\ (\dot{x} \\ & \underline{x\dot{x}+a\dot{x}} \\ & \quad\ \underline{-a\dot{x}} \\ & \quad x+a. \end{array}$$

Hence, the fluent is $x-a\times$ hyp. log. $\overline{a+x}$.

2. In the same manner, the fluent of $\frac{x\dot{x}}{a-x}=-x-a\times$ hyp. log. $\overline{a-x}$.

3. The fluent of $\frac{x\dot{x}}{x-a}=x+a\times$ hyp. log. $\overline{x-a}$. [illegible]

### (202.) FLUENT 2.

1. The fluent of $\frac{x^2\dot{x}}{a+x} = \frac{x^2}{2} - ax + a^2 \times$ hyp. log. $a+x$.

2. The fluent of $\frac{x^2\dot{x}}{a-x} = -\frac{x^2}{2} - ax - a^2 \times$ hyp. log. $\overline{a-x}$.

3. The fluent of $\frac{x^2\dot{x}}{x-a} = \frac{x^2}{2} + ax + a^2 \times$ hyp. log. $\overline{x-a}$.

### (203.) FLUENT 3.

The fluent of $\frac{px^3\dot{x}}{a-x}$ = the fluent of $p \times : -x^2\dot{x} - ax\dot{x} - a^2\dot{x} + \frac{a^3\dot{x}}{a-x} = -\frac{px^3}{3} - \frac{pax^2}{2} - pa^2x + pa^3 \times$ hyp. log. $\frac{a}{a-x}$; the last term being corrected by making $x = 0$. This fluent occurs in the problem for finding the content of the solid, generated by the revolution of the cissoid of Diocles about its axis.

### (204.) FLUENT 4.

1. To find the fluent of $\frac{x^m\dot{x}}{x-a}$, where $m$ is a whole positive number.

$$x-a)\,x^m\dot{x}\,(x^{m-1}\dot{x} + ax^{m-2}\dot{x} + \text{\&c.}$$
$$\underline{x^m\dot{x} - ax^{m-1}\dot{x}}$$
$$ax^{m-1}\dot{x}, \text{ \&c.}$$

Continue the division, till the index of $x$ in the remainder $= 0$; there are then $m$ terms in the quotient, the last of which is $a^{m-1}\dot{x}$, and the remainder $= \frac{a^m\dot{x}}{x-a}$. Hence, the whole fluent $= \frac{x^m}{m} + \frac{ax^{m-1}}{m-1} + \text{\&c.} + a^m \times$ hyp. log. $\overline{x-a}$.

2. The series is the same for the fluent of $\frac{x^m\dot{x}}{x+a}$, but the signs of the terms are alternately + and −.

3. The fluent of $\frac{x^m\dot{x}}{a-x}$ is found by dividing the numerator by $-x+a$; and the signs of the terms are all negative.

### (205.) FLUENT 5.

1. To find the fluent of $\frac{x^2\dot{x}}{a^2+x^2}$.

$$x^2+a^2)\, x^2\dot{x}\, (\dot{x}$$
$$\underline{x^2\dot{x} + a^2\dot{x}}$$
$$\underline{-\,a^2\dot{x}}$$
$$a^2 + x^2$$

Hence, the fluent $= x - a$ circular arc of radius $a$, and tangent $x$.

2. The fluent of $\frac{x^2\dot{x}}{a^2-x^2} = -x + \frac{a}{2} \times$ hyp. log. $\frac{a+x}{a-x}$.

3. The fluent of $\frac{x^2\dot{x}}{x^2-a^2} = x + \frac{a}{2} \times$ hyp. log. $\frac{x-a}{x+a}$.

4. In the same manner, the fluent of $\frac{x^4\dot{x}}{a^2 \pm x^2}$, &c. can be found.

### (206.) FLUENT 6.

1. To find the fluent of $\frac{x^m\dot{x}}{a^2+x^2}$, where $m$ is an even positive number.

The numerator divided by the denominator, in an inverted order, $= x^{m-2}\dot{x} - a^2x^{m-4}\dot{x} + a^4x^{m-6}\dot{x} - \&c. \pm a^{m-2}\dot{x} \mp \frac{a^m\dot{x}}{a^2+x^2}$, the number of terms in the quotient being $\frac{m}{2}$, and the remainder $\mp \frac{a^m\dot{x}}{a^2+x^2}$; hence, the fluent $= \frac{x^{m-1}}{m-1} - \frac{a^2x^{m-3}}{m-3} + \&c. \pm a^{m-2}x \mp a^{m-1} \times$ a circular arc of radius $a$, and tangent $x$.

2. In the same manner, the fluent of $\frac{x^m\dot{x}}{a^2-x^2}$, where $m$ is an even positive number, $= -\frac{x^{m-1}}{m-1} - \frac{a^2x^{m-3}}{m-3} - \&c. + \frac{a^{m-1}}{2} \times$ hyp. log. $\frac{a+x}{a-x}$.

3. The fluent of $\frac{x^m\dot{x}}{x^2-a^2}$, determined in the same manner, has all the terms positive; the remainder is $\frac{a^{m-1}}{2}$ × hyp. log. $\frac{x-a}{x+a}$.

## (207.) FLUENT 7.

1. To find the fluent of $\frac{x^3\dot{x}}{a^2+x^2}$.

The numerator divided by the denominator in an inverted order, gives $x\dot{x}-\frac{a^2x\dot{x}}{a^2+x^2}$; and the fluent $=\frac{x^2}{2}-\frac{a^2}{2}$ × hyp. log. $\overline{a^2+x^2}$.

2. The fluent of $\frac{x^3\dot{x}}{a^2-x^2}=-\frac{x^2}{2}-\frac{a^2}{2}$ × hyp. log. $\overline{a^2-x^2}$.

3. The fluent of $\frac{x^3\dot{x}}{x^2-a^2}=\frac{x^2}{2}+\frac{a^2}{2}$ × hyp. log. $\overline{x^2-a^2}$.

4. In the same manner, the fluents of $\frac{x^5\dot{x}}{a^2\pm x^2}$, $\frac{x^7\dot{x}}{a^2\pm x^2}$, &c. can be found.

## (208.) FLUENT 8.

1. To find the fluent of $\frac{x^m\dot{x}}{a^2+x^2}$, where $m$ is an odd number.

The numerator divided by the denominator $x^2+a^2$ gives $x^{m-2}\dot{x}-a^2x^{m-4}\dot{x}+a^4x^{m-6}\dot{x}-$ &c. $\pm a^{m-3}x\dot{x}$, the number of terms being $\frac{m-1}{2}$, and the remainder $\mp\frac{a^{m-1}x\dot{x}}{a^2+x^2}$; so that the fluent is $\frac{x^{m-1}}{m-1}-\frac{a^2x^{m-3}}{m-3}$ + &c., $\pm\frac{a^{m-3}x^2}{2}\mp$ the remainder $\frac{a^{m-1}}{2}$ × hyp. log. $a^2+x^2$.

2. The fluent of $\frac{x^m\dot{x}}{a^2-x^2}$ is obtained in the same manner; but the signs are all negative, and the remainder is $-\frac{a^{m-1}}{2}$ × hyp. log. $\overline{a^2-x^2}$.

3. The fluent of $\frac{x^m \dot{x}}{x^2 - a^2}$ has all the signs positive, and the remainder is $\frac{a^{m-1}}{2} \times$ hyp. log. $\overline{x^2 - a^2}$.

### (209.) FLUENT 9.

1. To find the fluent of $\frac{x^{rm-1}\dot{x}}{x^m - a^m}$, where $r$ is a whole positive number.

Divide the numerator by the denominator; the quotient is $x^{rm-m-1}\dot{x} + a^m x^{rm-2m-1}\dot{x} + a^{2m} x^{rm-3m-1}\dot{x} +$ &c. Let the division be continued for some number of terms as $p$; then the last term is $a^{m\times\overline{p-1}} x^{rm-pm-1}\dot{x}$, and the remainder is $\frac{a^{pm} x^{rm-pm-1}\dot{x}}{x^m - a^m}$; if the variable part of this remainder were $\frac{x^{m-1}\dot{x}}{x^m - a^m}$, the fluent could be found, and it is evident that the division may be continued till it is. Assume, therefore, $rm - pm - 1 = m - 1$; then $p = r - 1$, or the number of terms in the quotient is $r - 1$. Hence, the last term is $a^{m\times\overline{r-2}} x^{m-1}\dot{x}$, and the remainder is $\frac{a^{m\overline{r-1}} x^{m-1}\dot{x}}{x^m - a^m}$; therefore the required fluent $= \frac{x^{rm-m}}{rm - m} + \frac{a^m x^{rm-2m}}{rm - 2m} + \frac{a^{2m} x^{rm-3m}}{rm - 3m} +$ &c., $+ \frac{a^{m\times\overline{r-2}} x^m}{m} +$ the remainder $\frac{a^{rm-m}}{m} \times$ hyp. log. $\overline{x^m - a^m}$.

2. The fluent of $\frac{x^{rm-1}\dot{x}}{a^m + x^m}$ is found in the same manner. In this case, the terms are alternately positive and negative; in other respects the fluents are the same.

3. The fluent of $\frac{x^{rm-1}\dot{x}}{a^m - x^m}$ has all its terms negative.

4. The fluent of $\frac{x^{rm-1}\dot{x}}{a \sim bx^m}$, or, of $\frac{1}{b} \times \frac{x^{rm-1}\dot{x}}{\frac{a}{b} \sim x^m}$ is the

same with the preceding, $\frac{a}{b}$ being substituted in the result for $a^m$, and the whole series multiplied by the fraction $\frac{1}{b}$.

### (210.) FLUENT 10.

1. To find the fluent of $\frac{c-ry^2}{c+by^2} \times y\dot{y}$.

This quantity $= \frac{cy\dot{y}-ry^3\dot{y}}{c+by^2}$. Divide, in an inverted order,

$$\begin{array}{l} by^2+c\big)\,-ry^3\dot{y}+cy\dot{y}\big(\frac{-ry\dot{y}}{b} \\ \qquad\quad -ry^3\dot{y}-\frac{rcy\dot{y}}{b} \\ \hline \qquad\quad \frac{\overline{bc+rc}}{b} \times \frac{y\dot{y}}{by^2+c}. \end{array}$$

Hence the required fluent $= f.\frac{-ry\dot{y}}{b} + f.\frac{bc+rc}{b} \times \frac{y\dot{y}}{by^2+c}$ $= \frac{-ry^2}{2b} + \frac{bc+rc}{2b^2} \times$ hyp. log. $y^2 + \frac{c}{b}$.

2. If the fluxion be $\frac{ar^2-ry^2}{ar^2+\overline{a-r}.y^2} \times y\dot{y}$, $c=ar^2$, and $b=a-r$; therefore, the fluent is $\frac{-ry^2}{2\times\overline{a-r}} + \frac{\frac{1}{2}a^2r^2}{\overline{a-r}|^2} \times$ hyp. log. $y^2 + \frac{ar^2}{a-r}$.

3. The fluent of $\frac{ar^2+ry^2 \times y\dot{y}}{ar^2+\overline{a+r}.y^2\wedge} = \frac{ry^2}{2.\overline{a+r}} + \frac{\frac{1}{2}a^2r^2}{\overline{a+r}|^2} \times$ hyp. log. $y^2 + \frac{ar^2}{a+r}$.

## SECTION II.

RULE. Many fluents may be found by dividing the given fluxion into two parts, and considering it as a rectangle. For, since the fluxion of $xy = x\dot{y} + y\dot{x}$, the fluent of $x\dot{y} = xy - f.\ y\dot{x}$. In these cases, the variable quantity without the vinculum, is divisible by the fluxion of that under the vinculum.

### (211.) FLUENT 11.

To find the fluent of $-x^3\dot{x} \times \sqrt{a^2 - x^2}$.

Let $\sqrt{a^2 - x^2} \times -x\dot{x} = \dot{y}$, and $x^2 = z$; then the given fluxion $= z\dot{y}$; and its fluent $= zy - f.\ y\dot{z} = \frac{x^2}{3} \times \overline{a^2 - x^2}|^{\frac{3}{2}} - \frac{2}{3} \times f.\ x\dot{x} \times \overline{a^2 - x^2}|^{\frac{3}{2}} = \frac{x^2}{3} \times \overline{a^2 - x^2}|^{\frac{3}{2}} + \frac{2}{15} \times \overline{a^2 - x^2}|^{\frac{5}{2}} = \frac{x^2 \times \overline{a^2 - x^2}|^{\frac{3}{2}}}{3} + \frac{\overline{2a^2 - 2x^2}}{15} \times \overline{a^2 - x^2}|^{\frac{3}{2}} = \frac{3x^2 + 2a^2}{15} \times \overline{a^2 - x^2}|^{\frac{3}{2}}$.

### (212.) FLUENT 12.

Required the fluent of $\frac{x\dot{x}}{\sqrt{a+x}}$.

This $= x\dot{x} \times \overline{a+x}|^{-\frac{1}{2}}$. Let $\overline{a+x}|^{-\frac{1}{2}} \times \dot{x} = \dot{y}$; then $x\dot{y} =$ the proposed fluxion; and the fluent $= xy - f.\ y\dot{x} = \overline{a+x}|^{\frac{1}{2}} \times 2x - f.\ \overline{a+x}|^{\frac{1}{2}} \times 2\dot{x} = \overline{a+x}|^{\frac{1}{2}} \times 2x - \frac{4}{3} \times \overline{a+x}|^{\frac{3}{2}} = \overline{a+x}|^{\frac{1}{2}} \times \overline{2x - \frac{4a+4x}{3}} = \frac{\overline{a+x}|^{\frac{1}{2}} \times \overline{2x - 4a}}{3}$.

### (213.) FLUENT 13.

1. Required the fluent of $\frac{bx^{n-1}\dot{x}}{\sqrt{a+fx^n}}$.

Assume $\overline{a+fx^n}|^{-\frac{1}{2}} \times x^{n-1}\dot{x} = \dot{y}$, and $bx^n = z$. Then the fluent of $z\dot{y} = zy - f.\ y\dot{z} = bx^n \times \frac{2}{nf} \times \overline{a+fx^n}|^{\frac{1}{2}} - \frac{2b}{nf} \times f.\ \overline{a+fx^n}|^{\frac{1}{2}} \times nx^{n-1}\dot{x} = \frac{2b}{nf} \times \overline{a+fx^n}|^{\frac{1}{2}} \times x^n - \frac{4b}{3nf^2} \times \overline{a+fx^n}|^{\frac{3}{2}}$.

By the same Rule,

2. The fluent of $\frac{-x^3\dot{x}}{\sqrt{a^2-x^2}} = \overline{a^2-x^2}|^{\frac{1}{2}} \times \frac{\overline{2a^2+x^2}}{3}$; here assume $\overline{a^2-x^2}|^{-\frac{1}{2}} \times -x\dot{x} = \dot{y}$, and $x^2 = z$.

3. The fluent of $\frac{-x^5\dot{x}}{\overline{a^2-x^2}|^{\frac{3}{2}}} = \frac{x^4+4a^2x^2-8a^4}{3\times\sqrt{a^2-x^2}}$.

Here assume $\overline{a^2-x^2}|^{-\frac{3}{2}} \times -x\dot{x} = \dot{y}$, and $x^4 = z$.

4. The fluent of $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z} = \frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2} \times \overline{\frac{cz^n}{m+2} - \frac{a}{\overline{m+1}.\overline{m+2}}}$. Assume $\overline{a+cz^n}|^m \times z^{n-1}\dot{z} = \dot{y}$, and $dz^n = x$.

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## SECTION III.

### ON COMPARISON OF FLUENTS.

This is the method of deducing one fluent from another. It is not easy to lay down rules which will answer in every case where this method may be applied; but the general principle may be explained in the following manner.

Rule. Assume some quantity in the form of a rectangle $= p$; and of such a nature, that one part of its fluxion may be the fluxion of some known fluent, (as $A$); and the other part the fluxion ($\dot{N}$), whose fluent is required. Put this equal to $\dot{p}$; then since $\dot{N} = \pm\dot{p}\pm\dot{A}$, $N$ itself $= \pm p \pm A$.

The most general method of making this assumption, and which obtains in many of the following fluents, is this. Divide the quantity without the vinculum, by the fluxion of the quantity under the vinculum; omit the constant quantities, and put this quotient $= p$, having first increased the index of the whole quantity under the vinculum by 1; and proceed in the manner stated above. In other cases the method will best appear by the Examples.

### (214.) FLUENT 14.

Given the fluent of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ (Fluent Art 43.); required the fluent of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$.

Assume $x \times \sqrt{a^2+x^2}=p$; $\therefore \dot{x} \times \sqrt{a^2+x^2} + \frac{x\dot{x} \times x}{\sqrt{a^2+x^2}} = \dot{p}$; or, multiplying the numerator and denominator of the first part by $\sqrt{a^2+x^2}$, we have $\frac{a^2\dot{x}+x^2\dot{x}+x^2\dot{x}}{\sqrt{a^2+x^2}} = \dot{p}$; $\therefore \frac{x^2\dot{x}}{\sqrt{a^2+x^2}} = \frac{\dot{p}}{2} - \frac{a^2\dot{x}}{2 \times \sqrt{a^2+x^2}}$; and the fluent $= \frac{p}{2} - f\cdot \frac{a^2\dot{x}}{2\sqrt{a^2+x^2}} = \frac{x \times \sqrt{a^2+x^2}}{2} - \frac{1}{2}a^2 \times$ hyp. log. $x+\sqrt{a^2+x^2}$.

### (215.) FLUENT 15.

1. Given the fluent of $\frac{\dot{x}}{\sqrt{a^2-x^2}}$ (Fluent Art 44.); required the fluent of $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$.

Assume $x \times \sqrt{a^2-x^2}=p$; then, by taking the fluxion of this rectangle and reducing it, $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}} = \frac{a^2\dot{x}}{2\sqrt{a^2-x^2}} - \frac{\dot{p}}{2}$; and the required fluent $= \frac{1}{2}a \times$ a circular arc of radius (a,) and sine (x) $- \frac{x \times \sqrt{a^2-x^2}}{2}$.

This fluent is used in finding the area of the conchoid of Nicomedes.

2. By a similar process, the fluent of $\frac{x^2\dot{x}}{\sqrt{x^2-a^2}} = x \times \frac{\sqrt{x^2-a^2}}{2} + \frac{1}{2}a^2 \times$ hyp. log. $x+\sqrt{x^2-a^2}$.

### (216.) FLUENT 16.

1. Given the fluent of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$ $(\dot{A})$; to find the $f\cdot \frac{x^4\dot{x}}{\sqrt{a^2+x^2}}$.

Assume $x^3\times\sqrt{a^2+x^2}=p$; then, by taking the fluxion and reducing the expression, we have $\frac{x^4\dot{x}}{\sqrt{a^2+x^2}}=\frac{\dot{p}}{4}-\frac{3a^2\times\dot{A}}{4}$; ∴ the required fluent $=x^3\times\frac{\sqrt{a^2+x^2}}{4}-\frac{3a^2}{4}\times A$.

2. In the same manner, if the fluent of $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$ $(\dot{B})$ be given, the fluent of $\frac{x^4\dot{x}}{\sqrt{a^2-x^2}}=\frac{3a^2}{4}\times B-\frac{x^3\times\sqrt{a^2-x^2}}{4}$.

3. If the fluent of $\frac{x^2\dot{x}}{\sqrt{x^2-a^2}}$ $(\dot{C})$ be given, the fluent of $\frac{x^4\dot{x}}{\sqrt{x^2-a^2}}=\frac{x^3\times\sqrt{x^2-a^2}}{4}+\frac{3a^2}{4}\times C$.

4. And it is evident, that the fluents of $\frac{x^6\dot{x}}{\sqrt{x^2\pm a^2}}$, $\frac{x^8\dot{x}}{\sqrt{x^2\pm a^2}}$, $\frac{x^n\dot{x}}{\sqrt{x^2\pm a^2}}$, where $n$ is an even number, may be determined by knowing the fluent of the fluxion, which immediately precedes it in that series; and substituting $p$ as before. Thus, if the fluent of $\frac{x^{n-2}\dot{x}}{\sqrt{a^2+x^2}}=\dot{B}$ be given, the $f.\ \frac{x^n\dot{x}}{\sqrt{a^2+x^2}}=\frac{x^{n-1}\times\sqrt{a^2+x^2}}{n}-\frac{\overline{n-1}\,.\,a^2}{n}\times B$.

### (217.) FLUENT 17.

1. Given the fluent of $x\dot{x}\times\sqrt{a^2+x^2}$ $(\dot{D})$; to find the fluent of $\frac{x^3\dot{x}}{\sqrt{a^2+x^2}}$, where the index of $x$ is an odd number.

Assume $x^2\times\sqrt{a^2+x^2}=p$; then $2x\dot{x}\times\sqrt{a^2+x^2}+\frac{x^3\dot{x}}{\sqrt{a^2+x^2}}=\dot{p}$. In this instance it is not necessary to reduce the first part, as in the former cases; for we have at once $\frac{x^3\dot{x}}{\sqrt{a^2+x^2}}=\dot{p}-2\dot{D}$; ∴ the required fluent $=p-2D=x^2\times\sqrt{a^2+x^2}-2D$, where $D=\frac{\overline{a^2+x^2}|^{\frac{3}{2}}}{3}$.

2. In the same manner, the fluent of $\frac{x^3\dot{x}}{\sqrt{a^2-x^2}} = 2D - x^2 \times \sqrt{a^2-x^2}$, where $D = f.\, x\dot{x} \times \sqrt{a^2-x^2} = -\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$.

This fluent is used in finding the content of the solid generated by the revolution of the conchoid about its axis.

3. And the fluent of $\frac{x^3\dot{x}}{\sqrt{x^2-a^2}} = x^2 \times \sqrt{x^2-a^2} - 2D$, where $D = f.\, x\dot{x} \times \sqrt{x^2-a^2} = \frac{\overline{x^2-a^2}|^{\frac{3}{2}}}{3}$.

(218.) FLUENT 18.

1. Given the fluent of $\frac{x^3\dot{x}}{\sqrt{a^2+x^2}}$ $(\dot{E})$; to find $f.\, \frac{x^5\dot{x}}{\sqrt{a^2+x^2}}$.

Assume $x^4 \times \sqrt{a^2+x^2} = p$; then, by taking the fluxion and reducing it, $\frac{x^5\dot{x}}{\sqrt{a^2+x^2}} = \frac{\dot{p}}{5} - \frac{4a^2\dot{E}}{5}$; and the required fluent $= \frac{x^4 \times \sqrt{a^2+x^2}}{5} - \frac{4a^2}{5} \times E$.

2. Thus also, the fluent of $\frac{x^5\dot{x}}{\sqrt{a^2-x^2}}$, and of $\frac{x^5\dot{x}}{\sqrt{x^2-a^2}}$, may be found.

3. In the same manner, the fluent of $\frac{x^7\dot{x}}{\sqrt{a^2 \pm x^2}}$, $\frac{x^9\dot{x}}{\sqrt{a^2 \pm x^2}}$, &c. $\frac{x^n\dot{x}}{\sqrt{a^2 \pm x^2}}$ may be determined, where $n$ is an odd number, by knowing the fluent of the fluxion, which immediately precedes it in that series. Thus, if the fluent of $\frac{x^{n-2}\dot{x}}{\sqrt{a^2+x^2}} = \dot{E}$ be given, the fluent of $\frac{x^n\dot{x}}{\sqrt{a^2+x^2}} = \frac{x^{n-1} \times \sqrt{a^2+x^2}}{n} - \frac{\overline{n-1}\,.\,a^2}{n} \times E$.

## (219.) FLUENT 19.

1. Given the fluent of $\frac{\dot{x}}{\sqrt{a+x}}$ $(\dot{F})$; to find the fluent of $\frac{x\dot{x}}{\sqrt{a+x}}$.

Let $x \times \sqrt{a+x} = p$; then $\frac{a\dot{x} + \frac{3}{2}x\dot{x}}{\sqrt{a+x}} = \dot{p}$; and $\frac{x\dot{x}}{\sqrt{a+x}} = \frac{2\dot{p}}{3} - \frac{\frac{2}{3}a\dot{x}}{\sqrt{a+x}}$; and the fluent $= \frac{2p}{3} - \frac{2a}{3} \times F = \frac{2x \times \sqrt{a+x}}{3} - \frac{2a}{3} \times F$.

2. In the same manner we can find the fluent of $\frac{x\dot{x}}{\sqrt{a-x}}$, and the fluent of $\frac{x\dot{x}}{\sqrt{x-a}}$.

## (220.) FLUENT 20.

1 Given the fluent of $\frac{x\dot{x}}{\sqrt{a+x}}$ $(\dot{G})$; required the fluent of $\frac{x^2\dot{x}}{\sqrt{a+x}}$.

Assume $x^2 \times \sqrt{a+x} = p$; then, taking the fluxion and reducing it, $\frac{x^2\dot{x}}{\sqrt{a+x}} = \frac{2\dot{p}}{5} - \frac{4ax\dot{x}}{5 \times \sqrt{a+x}}$; and the fluent $= \frac{2x^2}{5} \times \sqrt{a+x} - \frac{4a \times G}{5}$.

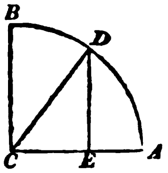
2. In the same manner, if the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{a+x}}$ $(\dot{G})$ be given, the fluent of $\frac{x^n\dot{x}}{\sqrt{a+x}} = \frac{2}{2n+1} \times x^n \times \sqrt{a+x} - \frac{2na}{2n+1} \times G$.

3. Thus also, the fluent of $\frac{x^n\dot{x}}{\sqrt{a \sim x}}$ may be found in terms of $p$, and the fluent, which precedes it, in the series.

4. The fluent of $\frac{x\dot{x}}{\sqrt{x-a}}$, by this method, $= \frac{2x \times \sqrt{x-a}}{3}$ $+ \frac{4a \times \sqrt{x-a}}{3}$.

### LEMMA.

If $AB$ be a quadrant, and $C$ the center of the circle, whose radius is $a$, and versed sine $x$, the right sine $DE = \sqrt{2ax-x^2}$. Hence, the fluxion of the area $AED = \dot{x} \times \sqrt{2ax-x^2}$; and the area $AED$ = the fluent of $\dot{x} \times \sqrt{2ax-x^2}$.



### (221.) FLUENT 21.

Given the fluent of $\dot{x} \times \sqrt{2ax-x^2}$ $(\dot{H})$; required the fluent of $x\dot{x} \times \sqrt{2ax-x^2}$.

Assume $\overline{2ax-x^2}|^{\frac{3}{2}} = p$; then $\frac{3}{2} \times \overline{2a\dot{x}-2x\dot{x}} \times \sqrt{2ax-x^2} = \dot{p}$; or, $3a\dot{x} \times \sqrt{2ax-x^2} - 3x\dot{x} \times \sqrt{2ax-x^2} = \dot{p}$; ∴ $x\dot{x} \times \sqrt{2ax-x^2} = a \times \dot{H} - \frac{\dot{p}}{3}$; and the required fluent $= a \times H - \frac{\overline{2ax-x^2}|^{\frac{3}{2}}}{3}$.

### (222.) FLUENT 22.

1. Given the fluent of $x\dot{x} \times \sqrt{2ax-x^2}$ $(\dot{I})$; required the fluent of $x^2\dot{x} \times \sqrt{2ax-x^2}$.

Assume $x \times \overline{2ax-x^2}|^{\frac{3}{2}} = p$; then, by taking the fluxion and reducing it as before, $5a\dot{I} - 4x^2\dot{x} \times \sqrt{2ax-x^2} = \dot{p}$; therefore the fluent of $x^2\dot{x} \times \sqrt{2ax-x^2} = \frac{5a}{4} \times I - \frac{x \times \overline{2ax-x^2}|^{\frac{3}{2}}}{4}$.

2. In the same manner the series may be continued; and if the fluent of $x^{n-1}\dot{x} \times \sqrt{2ax-x^2}$ $(\dot{I})$ be given, the fluent of $x^n\dot{x} \times \sqrt{2ax-x^2} = \frac{\overline{2n+1} \times aI - x^{n-1} \times \overline{2ax-x^2}|^{\frac{3}{2}}}{n+2}$.

(223.) FLUENT 23.

1. Given the fluent of $\frac{\dot{x}}{\sqrt{2ax+x^2}}$ ($\dot{K}$ Art. 43.); required the fluent of $\frac{x\dot{x}}{\sqrt{2ax+x^2}}$.

Assume $\sqrt{2ax+x^2}=p$; then $\frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}}=\dot{p}$; $\therefore \frac{x\dot{x}}{\sqrt{2ax+x^2}}=\dot{p}-a\dot{K}$; and the required fluent $=\sqrt{2ax+x^2}-a\times K$.

2. Thus also, if the fluent of $\frac{x\dot{x}}{\sqrt{2ax+x^2}}$ be given, the fluent of $\frac{x^2\dot{x}}{\sqrt{2ax+x^2}}$ may be found, by assuming $x\times\sqrt{2ax+x^2}=p$, and so on to any fluxion of this form. If the fluent of $x^{n-1}\dot{x}\times\sqrt{2ax+x^2}$ ($\dot{K}$) be given, by assuming $x^{n-}\times\sqrt{2ax+x^2}=p$, we have the fluent of $\frac{x^n\dot{x}}{\sqrt{2ax+x^2}}=\frac{x^{n-1}\times\sqrt{2ax+x^2}}{n}-\frac{\overline{2n-1}.a}{n}\times K$.

3. In the same way the fluent of $\frac{\dot{x}}{\sqrt{x^2-2ax}}$ (Art. 43.) being given, the fluent of $\frac{x\dot{x}}{\sqrt{x^2-2ax}}$ may be found; and the series may be continued to $\frac{x^n\dot{x}}{\sqrt{x^2-2ax}}$.

4. If the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{x^2-2ax}}$ ($\dot{K}$) be given, by assuming $x^{n-1}\times\sqrt{x^2-2ax}=p$, the fluent of $\frac{x^n\dot{x}}{\sqrt{x^2-2ax}}=\frac{x^{n-1}\times\sqrt{x^2-2ax}}{n}+\frac{\overline{2n-1}.a}{n}\times K$.

5. The fluent of $\frac{\dot{x}}{\sqrt{2ax-x^2}}$ being given (Art. 44.), that of $\frac{x\dot{x}}{\sqrt{2ax-x^2}}$, $\frac{x^2\dot{x}}{\sqrt{2ax-x^2}}$, &c., $\frac{x^n\dot{x}}{\sqrt{2ax-x^2}}$ may be obtained by the same method of continuation.

The fluent of $\frac{x\dot{x}}{\sqrt{ax-x^2}}$ found by this method = a circular arc of radius $\frac{1}{2}a$, and versed sine $x$, $-\sqrt{ax-x^2}$.

6. If the fluent of $\frac{x^{n-1}\dot{x}}{\sqrt{2ax-x^2}}$ $(\dot{K})$ be given, and $x^{n-1} \times \sqrt{2ax-x^2} = p$, the fluent of $\frac{x^n\dot{x}}{\sqrt{2ax-x^2}} = \frac{\overline{2n-1}}{n} \times aK - \frac{x^{n-1} \times \sqrt{2ax-x^2}}{n}$.

7. The fluent of $\frac{x^2\dot{x}}{\sqrt{ax-x^2}}$ by this method $= \frac{3a}{4} \times$ a circular arc of radius $\frac{1}{2}a$, and versed sine $x - \frac{x \times \sqrt{ax-x^2}}{2} - \frac{3a}{4} \times \sqrt{ax-x^2}$. This fluent is used in finding the area of the cissoid.

8. Thus also the fluent of $\dot{x} \times \sqrt{x-a}$ being given, the fluent of $x\dot{x} \times \sqrt{x-a}$, $x^2\dot{x} \times \sqrt{x-a}$, &c., $x^n\dot{x} \times \sqrt{x-a}$ can be found.

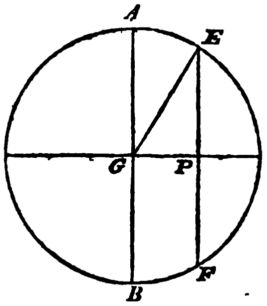
If the fluent of $x^{n-1}\dot{x} \times \sqrt{x-a}$ $(\dot{K})$ be given, by assuming $x^n \times \overline{x-a}\big|^{\frac{3}{2}} = p$, the fluent of $x^n\dot{x} \times \sqrt{x-a}$ is found $= \frac{2x^n \times \overline{x-a}\big|^{\frac{3}{2}}}{2n+3} + \frac{2na}{2n+3} \times K$.

9. And, given the fluxion of $\dot{x} \times \overline{x-a}\big|^m$, the fluent of $x\dot{x} \times \overline{x-a}\big|^m$, the fluent of $x^2\dot{x} \times \overline{x-a}\big|^m$, &c., $x^n\dot{x} \times \overline{x-a}\big|^m$ may be found.

10. If the fluent of $x^{n-1}\dot{x} \times \overline{x-a}\big|^m$ $(\dot{K})$ be given, by assuming $x^n \times \overline{x-a}\big|^{m+1} = p$, the fluent of $x^n\dot{x} \times \overline{x-a}\big|^m$ will be found $= \frac{x^n \times \overline{x-a}\big|^{m+1} + naK}{m+n+1}$.

11. By the same mode of continuation, the fluent of $x^n \dot{x} \times \sqrt{a \pm x}$, and of $x^n \dot{x} \times \overline{a \pm x}|^m$ may be found.

### Lemma.



If $GP = x$, and $GE = r$, $EF = 2 \times \sqrt{r^2 - x^2}$; and the fluxion of the area $ABFE = EF \times$ the fluxion of $GP = 2\dot{x} \times \sqrt{r^2 - x^2}$.

Cor. The fluent of $2\dot{x} \times \sqrt{r^2 - x^2} =$ the area $ABFE$.

### (224.) Fluent 24.

Given the fluent of $\dot{x} \times \sqrt{r^2 - x^2}$ $(A,)$ to find the fluent of $x^2 \dot{x} \times \sqrt{r^2 - x^2}$.

Assume $x \times \overline{r^2 - x^2}|^{\frac{3}{2}} = p$; then $\dot{x} \times \overline{r^2 - x^2}|^{\frac{3}{2}} - 3x^2 \dot{x} \times \sqrt{r^2 - x^2} = \dot{p}$; that, is $\overline{r^2 \dot{x} - 4x^2 \dot{x}} \times \sqrt{r^2 - x^2} = \dot{p}$; $\therefore x^2 \dot{x} \times \sqrt{r^2 - x^2} = \frac{r^2 \dot{x}}{4} \times \sqrt{r^2 - x^2} - \frac{\dot{p}}{4}$; and the fluent $= \frac{r^2 A - p}{4}$. When $x = 0$, $p = 0$; $\therefore$ no correction is wanted. Also, when $x = r$, $p = 0$; $\therefore$ the whole fluent between the values of $x = 0$ and $x = r$, is $\frac{r^2 A}{4} = \frac{r^2}{4} \times \frac{1}{2}$ the semi-circle $BCA = \frac{r^2}{16} \times$ the area of the whole circle $= \frac{r^2}{16} \times p r^2$, where $p = 3.14159$, &c.

Cor. 1. Hence, between similar values of $x$, the fluent of $4x^2 \dot{x} \times \sqrt{r^2 - x^2} = \frac{r^2}{4} \times p r^2$.

Cor. 2. Between the same values of $x$, the fluent of $4a^2 \dot{x} \times \sqrt{r^2 - x^2} + 4x^2 \dot{x} \times \sqrt{r^2 - x^2} = 2a^2 \times$ the area of the semi-circle $ACB$, $+ \frac{r^2}{4} \times pr^2 = \overline{a^2 + \frac{r^2}{4}} \times pr^2$.

This fluent occurs in Ex. 9. Chap. xi. where it is required to find the center of oscillation of a circle.

### (225.) FLUENT 25.

Given the fluent of $x^2\dot{x} \times \sqrt{r^2-x^2}$ $(B)$; to find the fluent of $x^4\dot{x} \times \sqrt{r^2-x^2}$ $(C)$.

Assume $x^3 \times \overline{r^2-x^2}|^{\frac{3}{2}} = p$; then $3x^2\dot{x} \times \overline{r^2-x^2} \times \sqrt{r^2-x^2} - 3x^4\dot{x} \times \sqrt{r^2-x^2} = \dot{p}$, or $3r^2\dot{B} - 6\dot{C} = \dot{p}$, $\therefore$ $C = \frac{3B-p}{6} = \frac{3r^2B - x^3 \times \overline{r^2-x^2}|^{\frac{3}{2}}}{6}$.

In the same manner if the fluent of $x^{n-2}\dot{x} \times \sqrt{r^2-x^2}$ $(\dot{D})$ be given, where $n$ is even, the fluent of $x^n\dot{x} \times \sqrt{r^2-x^2}$ can be found by assuming $x^{n-1} \times \overline{r^2-x^2}|^{\frac{3}{2}} = p$; the fluent $= \frac{\overline{n-1}.r^2D-p}{n+2}$.

COR. Hence the fluent of $\dot{x} \times \overline{r^2-x^2}|^{\frac{3}{2}}$, or the fluent of $r^2\dot{x} \times \sqrt{r^2-x^2} - f.\, x^2\dot{x} \times \sqrt{r^2-x^2} = r^2 \times \text{area } AGPE - \frac{r^2}{4} \times \text{area } AGPE + \frac{x \times \overline{r^2-x^2}|^{\frac{3}{2}}}{4} = \frac{3r^2}{4} \times \text{area } AGPE + \frac{x \times \overline{r^2-x^2}|^{\frac{3}{2}}}{4}$.

This fluent is used in Cotes' Problems, to compare the momentum of a sphere and circumscribing cylinder revolving round a common axis.

### (226.) FLUENT 26.

1. Given the fluent of $x\dot{x} \times \sqrt{a^2+x^2}$ $(\dot{L})$; to find the fluent of $x^3\dot{x} \times \sqrt{a^2+x^2}$.

Assume $x^2 \times \overline{a^2+x^2}|^{\frac{3}{2}} = p$; $\therefore$ $2x\dot{x} \times \overline{a^2+x^2} \times \sqrt{a^2+x^2} + 3x^3\dot{x} \times \sqrt{a^2+x^2} = \dot{p}$; $\therefore$ $2a^2\dot{L} + 2x^3\dot{x} \times \sqrt{a^2+x^2} + 3x^3\dot{x} \times \sqrt{a^2-x^2} = \dot{p}$; $\therefore$ $x^3\dot{x} \times \sqrt{a^2+x^2} = \frac{\dot{p}-2a^2\dot{L}}{5}$; and the required fluent $= \frac{x^2 \times \overline{a^2+x^2}|^{\frac{3}{2}} - 2a^2L}{5}$.

2. In the same manner the fluent of $x^5 \dot{x} \times \sqrt{a^2 \pm x^2}$, $x^7 \dot{x} \times \sqrt{a^2 \pm x^2}$, &c. may be found by means of the fluent which precedes it in that series.

### (227.) FLUENT 27.

1. In general, given the fluent of $x^{n-2} \dot{x} \times \sqrt{a^2+x^2}$, $(\dot{M})$; to find the fluent of $x^n \dot{x} \times \sqrt{a^2+x^2}$, where $n$ is an odd number.

Assume $x^{n-1} \times \overline{a^2+x^2}|^{\frac{3}{2}} = p$, then $\overline{n-1}.x^{n-2}\dot{x} \times \overline{a^2+x^2} \times \sqrt{a^2+x^2} + 3x^n \dot{x} \times \sqrt{a^2+x^2} = \dot{p}$, or $\overline{n-1}.a^2\dot{M} + \overline{n+2} \times x^n \dot{x} \sqrt{a^2+x^2} = \dot{p}$, & the fluent required $= \dfrac{x^{n-1} \times \overline{a^2+x^2}|^{\frac{3}{2}} - \overline{n-1}.a^2 M}{n+2}$.

2. The fluent of $x^n \dot{x} \times \sqrt{x^2 \mp a^2}$ may be found from similar data.

### (228.) FLUENT 28.

Given the fluent of $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$ $(\dot{L})$; to find the fluent of $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$ $(\dot{M})$.

Assume $\overline{a+cz^n}|^{m+1} \times dz_n = p$; then $\overline{m+1} \times ncz^{n-1}\dot{z} \times \overline{a+cz^n}|^m \times dz^n + ndz^{n-1}\dot{z} \times \overline{a+cz^n}|^{m+1} = \dot{p}$, or $\overline{m+1}.ncdz^{2n-1}\dot{z} \times \overline{a+cz^n}|^m + ndaz^{n-1}\dot{z} \times \overline{a+cz^n}|^m + ndcz^{2n-1}\dot{z} \times \overline{a+cz^n}|^m = \dot{p}$; that is, by collecting the quantities $\overline{m+2} \times nc\,\dot{M} + na\dot{L} = \dot{p}$; $\therefore M = \dfrac{p - naL}{\overline{m+2}.nc}$; or the required fluent $= \dfrac{\overline{a+cz^n}|^{m+1} \times dz^n}{\overline{m+2}.nc} - \dfrac{a}{\overline{m+2}.c} \times \dfrac{\overline{a+cz^n}|^{m+1} \times d}{\overline{m+1}.nc}$.

### (229.) FLUENT 29.

Given the fluent of $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$ $(\dot{M})$; required the fluent of $\overline{a+cz^n}|^m \times dz^{3n-1}\dot{z}$ $(\dot{N})$.

Assume $\overline{a+cz^n}|^{m+1} \times dz^{2n} = p$; then by taking the fluxion and reducing it, as in the last case, we get $\overline{m+3} \times nc \times \dot{N} = \dot{p}$

$-2an\dot{M}$; $\therefore N$, or the fluent required, $= \dfrac{p-2anM}{\overline{m+3}\,.\,nc} =$

$$\frac{\overline{a+cz^n}|^{m+1} \times dz^{2n}}{\overline{m+3}\,.\,nc} - \frac{2a}{\overline{m+3}\,.\,c} \times M.$$

In the same manner the series of fluents may be continued to $\overline{a+cz^n}|^{m} \times dz^{rn-1}\dot{z}$, where $r$ is a whole number.

For another method of finding this fluent, see Ex. 7[illegible] under Transformations.

### (230.) FLUENT 30.

Given the fluent of $\overline{a+cz^n}|^{m} \times z^{rn-1}\dot{z}$ ($\dot{P}$); to find the fluent of $\overline{a+cz^n}|^{m} \times z^{rn+n-1}\dot{z}$ ($\dot{Q}$).

Assume $\overline{a+cz^n}|^{m+1} \times z^{rn} = p$; $\therefore \overline{m+1}\,.\,ncz^{n-1}\dot{z} \times \overline{a+cz^n}|^{m} \times z^{rn} + nr \times \overline{a+cz^n} \times \overline{a+cz^n}|^{m} \times z^{rn-1}\dot{z} = \dot{p}$; that is, $\overline{m+1}\,.\,nc\,.\,\dot{Q} + nar\dot{P} + nrc\dot{Q} = \dot{p}\,\overline{m+r+1}\,.\,nc\,\dot{Q} = \dot{p} - nar\dot{P}$; $\therefore \dot{Q} = \dfrac{\overline{a+cz^n}|^{m+1} \times z^{rn} - narP}{\overline{m+r+1}\,.\,nc}$.

### (231.) FLUENT 31.

Given the fluent of $\dot{P}$ as before; to find the fluent of $\overline{a+cz^n}|^{m} \times z^{rn-n-1}\dot{z}$ ($\dot{R}$).

Assume $\overline{a+cz^n}|^{m+1} \times z^{rn-n} = p$; $\therefore \overline{m+1}\,.\,ncz^{n-1}\dot{z} \times \overline{a+cz^n}|^{m} \times z^{rn-n} + \overline{rn-n}\,.\,z^{rn-n-1}\dot{z} \times \overline{a+cz^n} \times \overline{a+cz^n}|^{m} = \dot{p}$; that is, $\overline{m+1}\,.\,nc\dot{P} + \overline{rn-n}\,.\,a \times \dot{R} + \overline{rn-n}\,.\,c\dot{P} = \dot{p}$; or, $\overline{m+r}\,.\,nc \times \dot{P} + \overline{r-1}\,.\,na\dot{R} = \dot{p}$; $\therefore R = \dfrac{p - \overline{m+r}\,.\,ncP}{\overline{r-1}\,.\,na}$

$$= \frac{\overline{a+cz^n}|^{m+1} \times z^{rn-n} - \overline{m+r}\,.\,ncP}{\overline{r-1}\,.\,na}.$$

### (232.) FLUENT 32.

Given the fluent of $\overline{a+cz^{n}}|^{m} \times z^{rn+n-1}\dot{z}$ $(\dot{S})$; to find the fluent of $\overline{a+cz^{n}}|^{m+1} \times z^{rn-1}\dot{z}$ $(\dot{T})$.

Assume $\overline{a+cz^{n}}|^{m+1} \times z^{rn}=p$; then $\overline{m+1}\,.\,ncz^{n-1}\dot{z} \times \overline{a+cz^{n}}|^{m} \times z^{rn}+nrz^{rn-1}\dot{z} \times \overline{a+cz^{n}}|^{m+1}=\dot{p}$; or, $\overline{m+1}\,.\,nc \times \dot{S}+n$[illegible]$\dot{T}=\dot{p}$; $\therefore T=\frac{p-\overline{m+1}\,.\,ncS}{nr}=\frac{\overline{a+cz^{n}}|^{m+1} \times z^{rn}-\overline{m+1}\,.\,ncS}{nr}$.

### (233.) FLUENT 33.

Given the fluent of $\overline{a+cz^{n}}|^{m} \times z^{rn-n-1}\dot{z}$ $(\dot{V})$; to find the fluent of $\overline{a+cz^{n}}|^{m-1} \times z^{rn-1}\dot{z}$ $(\dot{W})$.

Assume $\overline{a+cz^{n}}|^{m} \times z^{rn-n}=p$; $\therefore mcnz^{n-1}\dot{z} \times \overline{a+cz^{n}}|^{m-1} \times z^{rn-n}+\overline{rn-n}\,.\,z^{rn-n-1}\dot{z} \times \overline{a+cz^{n}}|^{m}=\dot{p}$; or, $mcn\dot{W}+\overline{rn-n} \times \dot{V}=\dot{p}$; $\therefore W=\frac{p-\overline{rn-n}\,.\,V}{mcn}=\frac{\overline{a+cz^{n}}|^{m} \times z^{rn-n}-\overline{rn-n}\,.\,V}{mcn}$.

### (234.) ● FLUENT 34.

Given the fluent of $\overline{a+cz^{n}}|^{m+1} \times z^{rn-1}\dot{z}$ $(\dot{X})$; to find the fluent of $\overline{a+cz^{n}}|^{m+1} \times z^{rn+n-1}\dot{z}$ $(\dot{Y})$.

Assume $\overline{a+cz^{n}}|^{m+2} \times z^{rn}=p$; $\therefore \overline{m+2}\,.\,ncz^{n-1}\dot{z} \times \overline{a+cz^{n}}|^{m+1} \times z^{rn}+rnz^{rn-1}\dot{z} \times \overline{a+cz^{n}} \times \overline{a+cz^{n}}|^{m+1}=\dot{p}$; or, $\overline{m+2}\,.\,nc\dot{Y}+rna\dot{X}+rnc\dot{Y}=\dot{p}$; $\therefore \overline{m+r+2}\,.\,nc \times \dot{Y}=\dot{p}-rna\dot{X}$; $\therefore Y=\frac{p-rnaX}{\overline{m+r+2}\,.\,nc}=\frac{\overline{a+cz^{n}}|^{m+2} \times z^{rn}-rnaX}{\overline{m+r+2}\,.\,nc}$.

### (235.) FLUENT 35.

Given the fluent of $\overline{a+cz^{n}}|^{m-1} \times z^{rn-1}\dot{z}$ $(\dot{Z})$; to find the fluent of $\overline{a+cz^{n}}|^{m-1} \times z^{rn-n-1}\dot{z}$ $(\dot{A})$.

Assume $\overline{a+cz^{n}}|^{m} \times z^{rn-n}=p$; $\therefore mncz^{n-1}\dot{z} \times \overline{a+cz^{n}}|^{m-1}$

$\times\ z^{rn-n} + \overline{rn-n} \times z^{rn-n-1}\dot{z} \times \overline{a+cz^n} \times \overline{a+cz^n}\rceil^{m-1} = \dot{p}$; or, $mnc\dot{Z} + \overline{rn-n}\,.\,a\dot{A} + \overline{rn-n}\,.\,c\dot{Z} = \dot{p}$; $\therefore A = \dfrac{p-\overline{m+r-1}\,.\,ncZ}{\overline{rn-n}\,.\,a} = \dfrac{\overline{a+cz^n}\rceil^{m} \times z^{rn-n} - \overline{m+r-1}\,.\,ncZ}{\overline{r-1}\,.\,na}$.

## (236.) FLUENT 36.

Given the fluent of $\overline{a+cz^n}\rceil^{m} \times z^r\dot{z}\ (\dot{B})$; to find the fluent of $\overline{a+cz^n}\rceil^{m} \times z^{r+n}\dot{z}\ (\dot{C})$.

Assume $\overline{a+cz^n}\rceil^{m+1} \times z^{r+1} = p$; then taking the fluxions, $\overline{m+1}\,.\,nc\,.\,\dot{C} + \overline{r+1}\,.\,a\dot{B} + \overline{r+1}\,.\,c\dot{C} = \dot{p}$; $\therefore \overline{mn+n+r+1} \times c\dot{C} = \dot{p} - \overline{r+1}\,.\,a\dot{B}$; $\therefore C = \dfrac{p-\overline{r+1}\,.\,aB}{\overline{mn+n+r+1}\,.\,e} = \dfrac{\overline{a+cz^n}\rceil^{m+1} \times z^{r+1} - \overline{r+1}\,.\,aB}{\overline{mn+n+r+1}\,.\,c}$.

## (237.) FLUENT 37.

Given the fluent of $\overline{a+cz^n}\rceil^{m} \times z^{r+n}$[illegible]$(\dot{C})$; to find the fluent of $\overline{a+cz^n}\rceil^{m+1} \times z^r\dot{z}\ (\dot{D})$.

Assume $\overline{a+cz^n}\rceil^{m+1} \times z^{r+1} = p$; then $\overline{m+1}\,.\,nc\dot{C} + \overline{r+1}\,.\,\dot{D} = \dot{p}$; $\therefore D = \dfrac{\overline{a+cz^n}\rceil^{m+1} \times z^{r+1} - \overline{m+1}\,.\,ncC}{r+1}$.

## (238.) FLUENT 38.

Given the fluent of $\overline{a+cz^n}\rceil^{m} \times z^r\dot{z}\ (\dot{A})$; to find the fluent of $\overline{a+cz^n}\rceil^{m} \times z^{r+n}\dot{z}\ (\dot{B})$, and of $\overline{a+cz^n}\rceil^{m+1} \times z^r\dot{z}\ (\dot{C})$.

Assume $\overline{a+cz^n}\rceil^{m+1} \times z^{r+1} = p$; then $\overline{m+1}\,.\,ncz^{r+n}\dot{z} \times \overline{a+cz^n}\rceil^{m} + \overline{r+1}\,.\,z^r\dot{z} \times \overline{a+cz^n}\rceil^{m+1} = \dot{p}$; or, $\overline{m+1}\,.\,nc\dot{B} + \overline{r+1}\,.\,\dot{C} = \dot{p}$; $\therefore B = \dfrac{p-\overline{r+1}\,.\,C}{\overline{m+1}\,.\,nc}$.

Also, $\dot{C} = z^r\dot{z} \times \overline{a+cz^n}\big|^{m+1} = az^r\dot{z} \times \overline{a+cz^n}\big|^{m} + cz^{r+n}\dot{z} \times \overline{a+cz^n}\big|^{m} = a\dot{A} + c\dot{B}$; $\therefore B = \frac{C-aA}{c}$. Hence $\frac{p-\overline{r+1}\,.\,C}{\overline{m+1}\,.\,nc} = \frac{C-aA}{c}$; $\therefore p-\overline{r+1}\,.\,C = \overline{m+1}\,.\,nC - \overline{m+1}\,.\,naA$; $\therefore C = \frac{p+\overline{m+1}\,..\,naA}{r+1+mn+n}$. Whence $B = \frac{C-aA}{c} = \frac{p+\overline{m+1}\,.\,naA}{r+1+mn+n \times c} - \frac{aA}{c}$.

Cor. By increasing $m$ each time by 1, and $r$ by $n$, the fluent may be continued as far as we please.

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## SECTION IV.

### ON THE TRANSFORMATION OF FLUXIONS.

**Rule I.** Many fluxions which are not of the common form, in Art. 39., may be reduced to that form by a transposition of the variable quantity, and the fluents found by that Rule.

(239.) **Fluent 39.**

To find the fluent of $\frac{\dot{x}}{x^2 \times \sqrt{a^2-x^2}}$.

Since $a^2 - x^2 = x^2 \times \overline{a^2x^{-2}-1}$; $\therefore \sqrt{a^2-x^2} = x \times \sqrt{a^2x^{-2}-1}$; hence $\frac{\dot{x}}{x^2 \times \sqrt{a^2-x^2}} = \frac{x^{-3}\dot{x}}{\sqrt{a^2x^{-2}-1}} = x^{-3}\dot{x} \times \overline{a^2x^{-2}-1}\big|^{-\frac{1}{2}}$; and the fluent $= \frac{2 \times \overline{a^2x^{-2}-1}\big|^{\frac{1}{2}}}{-2a^2} = -\frac{\sqrt{a^2-x^2}}{a^2x}$.

In the same manner the fluent of $\frac{\dot{x}}{x^2 \times \sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x}$.

### (240.) FLUENT 40.

To find the fluent of $\dfrac{a\dot{x}}{\overline{a^2+x^2}\big|^{\frac{3}{2}}}$.

Since $a^2+x^2=x^2\times\overline{a^2x^{-2}+1}$; $\therefore\ \dfrac{a\dot{x}}{\overline{a^2+x^2}\big|^{\frac{3}{2}}}=ax^{-3}\dot{x}\times$ $\overline{a^2x^{-2}+1}\big|^{-\frac{3}{2}}$; and the fluent $=\dfrac{\overline{a^2x^{-2}+1}\big|^{-\frac{1}{2}}}{a}=\dfrac{x}{a\times\sqrt{a^2+x^2}}$. This fluent is used in Prop. 1., Art. 135., on the Attraction of Bodies.

The fluent of $\dfrac{a\dot{x}}{\overline{a^2-x^2}\big|^{\frac{3}{2}}}=\dfrac{x}{a\times\sqrt{a^2-x^2}}$.

And by the same process, if $a=1$, we have the fluents of $\dfrac{\dot{x}}{\overline{1+x^2}\big|^{\frac{3}{2}}}=\dfrac{x}{\sqrt{1+x^2}}$, and of $\dfrac{\dot{x}}{\overline{1-x^2}\big|^{\frac{3}{2}}}=\dfrac{x}{\sqrt{1-x^2}}$.

### (241.) FLUENT 41.

To find the fluent of $\dfrac{z^{-\frac{1}{2}n-1}\dot{z}}{\sqrt{a+cz^n}}$.

Since $a+cz^n=z^n\times\overline{az^{-n}+c}$; $\therefore\ \sqrt{a+cz^n}=z^{\frac{n}{2}}\times\sqrt{az^{-n}+c}$, and the given fluxion $=z^{-n-1}\dot{z}\times\overline{az^{-n}+c}\big|^{-\frac{1}{2}}$; and the fluent $=-\dfrac{2}{na}\times\dfrac{\sqrt{a+cz^n}}{z^{\frac{n}{2}}}$.

The fluent of $\dfrac{z^{-\frac{1}{2}n-1}\dot{z}}{\sqrt{a-cz^n}}=-\dfrac{2}{na}\times\dfrac{\sqrt{a-cz^n}}{z^{\frac{n}{2}}}$.

### (242.) FLUENT 42.

To find the fluent of $\dfrac{\dot{x}\times\sqrt{a^2+x^2}}{x^4}$.

Since $a^2+x^2=x^2\times\overline{a^2x^{-2}+1}$; $\therefore\ \dfrac{\dot{x}\times\sqrt{a^2+x^2}}{x^4}=x^{-3}\dot{x}\times$ $\sqrt{a^2x^{-2}+1}$; and the fluent $=-\dfrac{1}{3a^2x^3}\times\overline{a^2+x^2}\big|^{\frac{3}{2}}$.

**Rule II.** If the index of the variable quantity without the vinculum increased by 1 be some aliquot part of the corresponding index under the vinculum, substitute for that power of the variable quantity, which is obtained by dividing the index under the vinculum by the number which expresses the aliquot part.

### (243.) Fluent 43.

1. To find the fluent of $\frac{x\dot{x}}{\sqrt{a^4-x^4}}$.

Let $y=x^2$, and $a^4=b^2$; then the given fluxion $=\frac{\frac{1}{2}\dot{y}}{\sqrt{b^2-y^2}}$, and the fluent $=\frac{1}{2b}\times$ circular arc of radius $b$, and sine $y$.

2. In the same manner the fluent of $\frac{x\dot{x}}{\sqrt{a^4+x^4}}=\frac{1}{2}$ hyp. log. $y+\sqrt{b^2+y^2}$.

3. And the fluent of $\frac{x\dot{x}}{\sqrt{x^4-a^4}}=\frac{1}{2}$ hyp. log. $y+\sqrt{y^2-b^2}$.

### (244.) Fluent 44.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n\pm z^n}}$.

1. Assume $c^{\frac{n}{2}}=a$; and $z^{\frac{n}{2}}=x$; then $z^{\frac{1}{2}n-1}\dot{z}=\frac{2\dot{x}}{n}$, and the fluxion $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n+z^n}}$ is transformed into $\frac{\frac{2}{n}\dot{x}}{\sqrt{a^2+x^2}}$; of which the fluent is $\frac{2}{n}\times$ hyp. log. $\overline{x+\sqrt{a^2+x^2}}$.

(240.) **FLUENT 40.**

To find the fluent of $\dfrac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$.

Since $a^2+x^2 = x^2 \times \overline{a^2x^{-2}+1}$; $\therefore \dfrac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}} = ax^{-3}\dot{x} \times \overline{a^2x^{-2}+1}|^{-\frac{3}{2}}$; and the fluent $= \dfrac{\overline{a^2x^{-2}+1}|^{-\frac{1}{2}}}{a} = \dfrac{x}{a \times \sqrt{a^2+x^2}}$. This fluent is used in Prop. 1., Art. 135., on the Attraction of Bodies.

The fluent of $\dfrac{a\dot{x}}{\overline{a^2-x^2}|^{\frac{3}{2}}} = \dfrac{x}{a \times \sqrt{a^2-x^2}}$.

And by the same process, if $a=1$, we have the fluents of $\dfrac{\dot{x}}{\overline{1+x^2}|^{\frac{3}{2}}} = \dfrac{x}{\sqrt{1+x^2}}$, and of $\dfrac{\dot{x}}{\overline{1-x^2}|^{\frac{3}{2}}} = \dfrac{x}{\sqrt{1-x^2}}$.

(241.) **FLUENT 41.**

To find the fluent of $\dfrac{z^{-\frac{1}{2}n-1}\dot{z}}{\sqrt{a+cz^n}}$.

Since $a+cz^n = z^n \times \overline{az^{-n}+c}$; $\therefore \sqrt{a+cz^n} = z^{\frac{n}{2}} \times \sqrt{az^{-n}+c}$, and the given fluxion $= z^{-n-1}\dot{z} \times \overline{az^{-n}+c}|^{-\frac{1}{2}}$; and the fluent $= -\dfrac{2}{na} \times \dfrac{\sqrt{a+cz^n}}{z^{\frac{n}{2}}}$.

The fluent of $\dfrac{z^{-\frac{1}{2}n-1}\dot{z}}{\sqrt{a-cz^n}} = -\dfrac{2}{na} \times \dfrac{\sqrt{a-cz^n}}{z^{\frac{n}{2}}}$.

(242.) **FLUENT 42.**

To find the fluent of $\dfrac{\dot{x} \times \sqrt{a^2+x^2}}{x^4}$.

Since $a^2+x^2 = x^2 \times \overline{a^2x^{-2}+1}$; $\therefore \dfrac{\dot{x} \times \sqrt{a^2+x^2}}{x^4} = x^{-3}\dot{x} \times \sqrt{a^2x^{-2}+1}$; and the fluent $= -\dfrac{1}{3a^2x^3} \times \overline{a^2+x^2}|^{\frac{3}{2}}$.

**Rule II.** If the index of the variable quantity without the vinculum increased by 1 be some aliquot part of the corresponding index under the vinculum, substitute for that power of the variable quantity, which is obtained by dividing the index under the vinculum by the number which expresses the aliquot part.

### (243.) Fluent 43.

1. To find the fluent of $\frac{x\dot{x}}{\sqrt{a^4 - x^4}}$.

Let $y = x^2$, and $a^4 = b^2$; then the given fluxion $= \frac{\frac{1}{2}\dot{y}}{\sqrt{b^2 - y^2}}$, and the fluent $= \frac{1}{2b} \times$ circular arc of radius $b$, and sine $y$.

2. In the same manner the fluent of $\frac{x\dot{x}}{\sqrt{a^4 + x^4}} = \frac{1}{2}$ hyp. log. $y + \sqrt{b^2 + y^2}$.

3. And the fluent of $\frac{x\dot{x}}{\sqrt{x^4 - a^4}} = \frac{1}{2}$ hyp. log. $y + \sqrt{y^2 - b^2}$.

### (244.) Fluent 44.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n \pm z^n}}$.

1. Assume $c^{\frac{n}{2}} = a$; and $z^{\frac{n}{2}} = x$; then $z^{\frac{1}{2}n-1}\dot{z} = \frac{2\dot{x}}{n}$, and the fluxion $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n + z^n}}$ is transformed into $\frac{\frac{2}{n}\dot{x}}{\sqrt{a^2 + x^2}}$; of which the fluent is $\frac{2}{n} \times$ hyp. log. $\overline{x + \sqrt{a^2 + x^2}}$.

### (240.) FLUENT 40.

To find the fluent of $\frac{a\dot{x}}{\overline{a^2+x^2}\vert^{\frac{3}{2}}}$.

Since $a^2+x^2=x^2\times\overline{a^2x^{-2}+1}$; $\therefore\ \frac{a\dot{x}}{\overline{a^2+x^2}\vert^{\frac{3}{2}}}=ax^{-3}\dot{x}\times$ $\overline{a^2x^{-2}+1}\vert^{-\frac{3}{2}}$; and the fluent $=\frac{\overline{a^2x^{-2}+1}\vert^{-\frac{1}{2}}}{a}=\frac{x}{a\times\sqrt{a^2+x^2}}$. This fluent is used in Prop. 1., Art. 135., on the Attraction of Bodies.

The fluent of $\frac{a\dot{x}}{\overline{a^2-x^2}\vert^{\frac{3}{2}}}=\frac{x}{a\times\sqrt{a^2-x^2}}$.

And by the same process, if $a=1$, we have the fluents of $\frac{\dot{x}}{\overline{1+x^2}\vert^{\frac{3}{2}}}=\frac{x}{\sqrt{1+x^2}}$, and of $\frac{\dot{x}}{\overline{1-x^2}\vert^{\frac{3}{2}}}=\frac{x}{\sqrt{1-x^2}}$.

### (241.) FLUENT 41.

To find the fluent of $\frac{z^{-\frac{1}{2}n-1}\dot{z}}{\sqrt{a+cz^n}}$.

Since $a+cz^n=z^n\times\overline{az^{-n}+c}$; $\therefore\ \sqrt{a+cz^n}=z^{\frac{n}{2}}\times\sqrt{az^{-n}+c}$, and the given fluxion $=z^{-n-1}\dot{z}\times\overline{az^{-n}+c}\vert^{-\frac{1}{2}}$; and the fluent $=-\frac{2}{na}\times\frac{\sqrt{a+cz^n}}{z^{\frac{n}{2}}}$.

The fluent of $\frac{z^{-\frac{1}{2}n-1}\dot{z}}{\sqrt{a-cz^n}}=-\frac{2}{na}\times\frac{\sqrt{a-cz^n}}{z^{\frac{n}{2}}}$.

### (242.) FLUENT 42.

To find the fluent of $\frac{\dot{x}\times\sqrt{a^2+x^2}}{x^4}$.

Since $a^2+x^2=x^2\times\overline{a^2x^{-2}+1}$; $\therefore\ \frac{\dot{x}\times\sqrt{a^2+x^2}}{x^4}=x^{-3}\dot{x}\times$ $\sqrt{a^2x^{-2}+1}$; and the fluent $=-\frac{1}{3a^2x^3}\times\overline{a^2+x^2}\vert^{\frac{3}{2}}$.

**Rule II.** If the index of the variable quantity without the vinculum increased by 1 be some aliquot part of the corresponding index under the vinculum, substitute for that power of the variable quantity, which is obtained by dividing the index under the vinculum by the number which expresses the aliquot part.

### (243.) Fluent 43.

1. To find the fluent of $\frac{x\dot{x}}{\sqrt{a^4-x^4}}$.

Let $y=x^2$, and $a^4=b^2$; then the given fluxion $=\frac{\frac{1}{2}\dot{y}}{\sqrt{b^2-y^2}}$, and the fluent $=\frac{1}{2b}\times$ circular arc of radius $b$, and sine $y$.

2. In the same manner the fluent of $\frac{x\dot{x}}{\sqrt{a^4+x^4}}=\frac{1}{2}$ hyp. log. $y+\sqrt{b^2+y^2}$.

3. And the fluent of $\frac{x\dot{x}}{\sqrt{x^4-a^4}}=\frac{1}{2}$ hyp. log. $y+\sqrt{y^2-b^2}$.

### (244.) Fluent 44.

To find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n\pm z^n}}$.

1. Assume $c^{\frac{n}{2}}=a$; and $z^{\frac{n}{2}}=x$; then $z^{\frac{1}{2}n-1}\dot{z}=\frac{2\dot{x}}{n}$, and the fluxion $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n+z^n}}$ is transformed into $\frac{\frac{2}{n}\dot{x}}{\sqrt{a^2+x^2}}$; of which the fluent is $\frac{2}{n}\times$ hyp. log. $\overline{x+\sqrt{a^2+x^2}}$.

### (253.) FLUENT 52.

To find the fluent of $\frac{\dot{x}}{\overline{a+x}|^2 \times \sqrt{a^2-x^2}}$.

Let $\frac{1}{a+x} = y$; $\therefore$ $-\dot{y} = \frac{\dot{x}}{\overline{a+x}|^2}$; and $a+x = \frac{1}{y}$; $\therefore$ $x = \frac{1}{y} - a$ and $x^2 = \frac{1}{y^2} - \frac{2a}{y} + a^2$; $\therefore$ $\sqrt{a^2-x^2} = \sqrt{\frac{2a}{y} - \frac{1}{y^2}} = \frac{\sqrt{2ay-1}}{y}$; $\therefore$ the fluxion $= \frac{-y\dot{y}}{\sqrt{2ay-1}} = \frac{-y\dot{y}}{\sqrt{2a} \times \sqrt{y - \frac{1}{2a}}}$, which is of the same form with Fluent 19.

### (254.) FLUENT 53.

To find the fluent of $\frac{z\dot{z}}{\overline{a+z}|^3 \times \sqrt{a^2+az+z^2}}$.

Let $x = \frac{a^2}{a+z}$; then $z = \frac{a^2-ax}{x} = a \times \frac{a-x}{x}$; $\therefore$ $\dot{z} = -\frac{a^2\dot{x}}{x^2}$; $z\dot{z} = -\frac{a^3\dot{x} \times \overline{a-x}}{x^3}$; $\sqrt{a^2+az+z^2} = \frac{a}{x} \times \sqrt{a^2-ax+x^2}$; $\therefore$ the given fluxion $= \frac{x^2\dot{x} - ax\dot{x}}{a^4 \times \sqrt{a^2-ax+x^2}}$, of which the fluent is found in Fluents 82, 83.

In some cases it is useful to substitute not for the reciprocal of the denominator, but for the denominator itself.

### (255.) FLUENT 54.

To find the fluent of $\frac{Lx\dot{x}+M\dot{x}}{\overline{x-p}|^2}$.

Let $x-p = z$; $x = z+p$, and $\dot{x} = \dot{z}$; hence the given fluxion $= \frac{Lz\dot{z} + Lp\dot{z} + M\dot{z}}{z^2} =$ (if $Lp+M = b$) $\frac{Lz\dot{z} + b\dot{z}}{z^2} =$

$\frac{L\dot{z}}{z} + \frac{b\dot{z}}{z^2}$; whose fluent is $L \times$ hyp. log. $z - \frac{b}{z} = L \times$ hyp. log. $\overline{x-p} - \frac{b}{x-p}$.

### (256.) FLUENT 55.

To find the fluent of $\frac{az\dot{z} - 3az^3\dot{z}}{\overline{1+z^2}|^3}$.

Let $1 + z^2 = x$; then $z^2 = x - 1$, and $z^4 = x^2 - 2x + 1$; $\therefore z^3\dot{z} = \frac{x\dot{x}}{2} - \frac{\dot{x}}{2}$; also $z\dot{z} = \frac{\dot{x}}{2}$; and $\overline{1+z^2}|^3 = x^3$; $\therefore$ the given fluxion $= \frac{\frac{1}{2}a\dot{x} - \frac{3a}{2} \times x\dot{x} + \frac{3a}{2} \times \dot{x}}{x^3} = \frac{2a\dot{x}}{x^3} - \frac{3a\dot{x}}{2x^2}$;

and the fluent $= \frac{3a}{2x} - \frac{a}{x^2} = \frac{3a}{2 \times \overline{1+z^2}} - \frac{a}{\overline{1+z^2}|^2}$.

**Rule 4.** The given fluxion may frequently be reduced to a better form by actual multiplication or division.

### (257.) FLUENT 56.

The fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a+x}} = \frac{f.\ x\dot{x}}{\sqrt{ax+x^2}}$, $x^{\frac{1}{2}}$ being multiplied both into the numerator and denominator; and the fluent (Ex. 23.) $= \sqrt{ax+x^2} - \frac{1}{2}a \times$ hyp. log. $x + \frac{a}{2} + \sqrt{x^2+ax}$.

### (258.) FLUENT 57.

The fluent of $\frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a+x}} = \frac{f.\ x^2\dot{x}}{\sqrt{ax+x^2}}$, which corresponds with Fluent 23.

### (259.) FLUENT 58.

1. The fluent of $\dot{x} \times \sqrt{x^2 \pm a^2}$ = the fluent of $\frac{x^2\dot{x} \pm a^2\dot{x}}{\sqrt{x^2 \pm a^2}}$;

the fluent of the first part is given in Fluent 14., and of the last in Art. 43.

2. The fluents of $x^2\dot{x} \times \sqrt{a^2 \pm x^2}$, &c.; $x^n\dot{x} \times \sqrt{a^2 \pm x^2}$, where $n$ is an even number may be found in the same way.

3. The fluent of $\frac{\dot{y} \times \sqrt{y^2 + b^2}}{b}$ by this method $= \frac{1}{b} \times$ $\frac{f.\, y^2\dot{y}}{\sqrt{y^2+b^2}} + \frac{b^2\dot{y}}{\sqrt{y^2+b^2}} = \frac{1}{b} \times :\ \frac{y \times \sqrt{b^2+y^2}}{2} - \frac{1}{2}b^2 \times$ hyp. log. $y+\sqrt{b^2+y^2}+b^2 \times$ hyp. log. $y+\sqrt{b^2+y^2} = \frac{1}{2b} \times \sqrt{y^4+b^2y^2} +$ $\frac{1}{2}b \times$ hyp. log. $y+\sqrt{b^2+y^2}$. This fluent is used in finding the length of the common parabola, Ex. 2. Art. 55; of the Spiral of Archimedes, Prob. 6. Art. 124; and of the Surface of a Solid generated by the revolution of the logarithmic curve about its axis, Art. 133, Prob. 6.

### (260.) FLUENT 59.

The fluent of $\frac{\dot{x} \times \sqrt{2ax+x^2}}{x} = \frac{f.\, 2a\dot{x}}{\sqrt{2ax+x^2}} + \frac{f.\, x\dot{x}}{\sqrt{2ax+x^2}}$; for which see Art. 43., and Fluent 23.

Thus also the fluents of $\frac{\dot{x} \times \sqrt{2ax-x^2}}{x}$ and of $\frac{\dot{x} \times \sqrt{x^2-2ax}}{x}$ by Fluent 23.

### (261.) FLUENT 60.

1. The fluent of $\frac{\dot{x} \times \sqrt{a^2+x^2}}{x} = \frac{f.\, a^2\dot{x}}{x\sqrt{a^2+x^2}} + \frac{x\dot{x}}{\sqrt{a^2+x^2}} =$ $\frac{a}{2} \times$ hyp. log. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a} + \sqrt{a^2+x^2}$.

This fluent is used in finding the length of the logarithmic curve, Art. 133. Prob. 5.

2. The fluent of $\frac{\dot{x} \times \sqrt{x^2-a^2}}{x} = \frac{f.\, x\dot{x}}{\sqrt{x^2-a^2}} - \frac{f.\, a^2\dot{x}}{x \times \sqrt{x^2-a^2}}$ $= \sqrt{x^2-a^2} -$ a circular arc of radius $a$, and secant $x$.

### (262.) FLUENT 61.

The fluent of $\frac{a\dot{x}}{x^{\frac{1}{2}}\times\sqrt{a+x}} = f.\frac{a\dot{x}}{\sqrt{ax+x^2}} = a\times$ hyp. log. $x+\frac{a}{2}+\sqrt{x^2+ax}$.

RULE 5. The given fluxion may frequently be reduced to a better form, by the addition and subtraction of another quantity, or by dividing it into other parts.

### (263.) FLUENT 62.

To find the fluent of $\frac{x\dot{x}}{a\pm x}$.

1. $\frac{x\dot{x}}{a+x}=\dot{x}+\frac{x\dot{x}}{a+x}-\dot{x}=\dot{x}-\frac{a\dot{x}}{a+x}$; ∴ the $f.\frac{x\dot{x}}{a+x}=x-a\times$ hyp. log. $\overline{a+x}$.

2. $\frac{x\dot{x}}{a-x}=\dot{x}+\frac{x\dot{x}}{a-x}-\dot{x}=\frac{a\dot{x}}{a-x}-\dot{x}$; ∴ $f.\frac{x\dot{x}}{a-x}=-x-a\times$ hyp. log. $\overline{a-x}$.

3. $\frac{x\dot{x}}{x-a}=\dot{x}+\frac{x\dot{x}}{x-a}-\dot{x}=\dot{x}+\frac{a\dot{x}}{x-a}$; ∴ $f.\frac{x\dot{x}}{x-a}=x+a\times$ hyp. log. $\overline{x-a}$.

### (264.) FLUENT 63.

To find the fluent of $\frac{x^2\dot{x}}{a^2\pm x^2}$.

1. $\frac{x^2\dot{x}}{a^2+x^2}=\dot{x}+\frac{x^2\dot{x}}{a^2+x^2}-\dot{x}=\dot{x}-\frac{a^2\dot{x}}{a^2+x^2}$; and the fluent = $x$ - a circular arc of radius $a$, and tangent $x$.

2. In the same manner, the fluent $\frac{x^2\dot{x}}{a^2-x^2}=\frac{a}{2}\times$ hyp. log. $\frac{a+x}{a-x}-x$.

3. And the fluent $\frac{x^2\dot{x}}{x^2-a^2}=x+\frac{a}{2}\times$ hyp. log. $\overline{\frac{x-a}{x+a}}$.

### (265.) FLUENT 64.

To find the fluent $\frac{x^3\dot{x}}{a^2\pm x^2}$.

1. $\frac{x^3\dot{x}}{a^2+x^2}=x\dot{x}+\frac{x^3\dot{x}}{a^2+x^2}-x\dot{x}=x\dot{x}-\frac{a^2x\dot{x}}{a^2+x^2}$; $\therefore f.\frac{x^3\dot{x}}{a^2+x^2}$ $=\frac{x^2}{2}-\frac{a^2}{2}\times$ hyp. log. $a^2+x^2$.

2. The $f.\frac{x^3\dot{x}}{a^2-x^2}$ in the same manner $=-\frac{x^2}{2}-\frac{a^2}{2}\times$ hyp. log. $\overline{a^2-x^2}$.

3. And $f.\frac{x^3\dot{x}}{x^2-a^2}=\frac{x^2}{2}+\frac{a^2}{2}\times$ hyp. log. $\overline{x^2-a^2}$.

### (266.) FLUENT 65.

To find the fluent of $\frac{\dot{x}}{x}\times\frac{1}{1-x}$.

1. This $=\frac{\dot{x}}{x-x^2}=\frac{\dot{x}-2x\dot{x}}{x-x^2}+\frac{2x\dot{x}}{x-x^2}=\frac{\dot{x}-2x\dot{x}}{x-x^2}+\frac{2\dot{x}}{1-x}$; and the fluent = hyp. log. $\overline{x-x^2}-2\times$ hyp. log. $\overline{1-x}=$ hyp. log. $\overline{x-x^2}-$ hyp. log. $\overline{1-x}\rvert^2$.

2. In the same manner, the $f.\frac{\dot{x}}{x}\times\frac{1}{1+x}=$ hyp. log. $\overline{x+x^2}-$ hyp. log. $\overline{1+x}\rvert^2$.

### (267.) FLUENT 66.

To find the fluent of $\frac{\dot{x}\times\sqrt{2ax+x^2}}{x}$.

This $=\frac{2ax\dot{x}}{x\times\sqrt{2ax+x^2}}+\frac{x\dot{x}}{\sqrt{2ax+x^2}}=\frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}}+\frac{a\dot{x}}{\sqrt{2ax+x^2}}$; and the fluent $=\sqrt{2ax+x^2}+a\times$ hyp. log. of $x+a+\sqrt{2ax+x^2}$ (Art. 43).

For another method, see Fluent 59.

### (268.) FLUENT 67.

The fluent of $y^{-\frac{1}{2}}\dot{y} \times \sqrt{a-y} = f.\,\frac{a\dot{y}-y\dot{y}}{\sqrt{ay-y^2}} = f.\,\frac{\frac{1}{2}a\dot{y}-y\dot{y}}{\sqrt{ay-y^2}} + f.\,\frac{\frac{1}{2}a\dot{y}}{\sqrt{ay-y^2}} = \sqrt{ay-y^2}$ + a circular arc of radius $\frac{1}{2}a$, and versed sine $y$.

### (269.) FLUENT 68.

The fluent of $\dot{y} \times \sqrt{\frac{y}{a-y}} = f.\,\frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}} = f.\,\frac{y\dot{y}}{\sqrt{ay-y^2}} = -1 \times f.\,\frac{\frac{1}{2}a\dot{y}-y\dot{y}}{\sqrt{ay-y^2}} + f.\,\frac{\frac{1}{2}a\dot{y}}{\sqrt{ay-y^2}} = -\sqrt{ay-y^2}$ + a circular arc of radius $\frac{1}{2}a$, and versed sine $y$.

### (270.) FLUENT 69.

The fluent of $\frac{x\dot{x}}{\sqrt{ax+x^2}} = f.\,\frac{\frac{1}{2}a\dot{x}+x\dot{x}}{\sqrt{ax+x^2}} - f.\,\frac{\frac{1}{2}a\dot{x}}{\sqrt{ax+x^2}} = \sqrt{ax+x^2} - \frac{1}{2}a \times \text{hyp. log. } x + \frac{a}{2} + \sqrt{x^2+ax}.$

### (271.) FLUENT 70.

The fluent of $\frac{x\dot{x}}{a^2+bx} = f.\,\frac{1}{b} \times \frac{x\dot{x}}{c+x} = \left(\text{if } c = \frac{a^2}{b}\right),\ \frac{1}{b} \times f.\,\frac{c\dot{x}+x\dot{x}-c\dot{x}}{c+x} = \frac{1}{b} \times f.\,\dot{x} - f.\,\frac{c\dot{x}}{c+x} = \frac{1}{b} \times \overline{x - c \times \text{hyp. log. } \overline{c+x}} = \frac{1}{b} \times x - \frac{a^2}{b^2} \times \text{hyp. log. } \frac{a^2}{b} + x.$

### (272.) FLUENT 71.

1. The fluent of $\frac{\dot{x}}{\overline{2x-x^2}|^{\frac{3}{2}}} = f.\,\frac{\dot{x}-x\dot{x}}{\overline{2x-x^2}|^{\frac{3}{2}}} + f.\,\frac{x\dot{x}}{\overline{2x-x^2}|^{\frac{3}{2}}} = f.\,\overline{\dot{x}-x\dot{x}} \times \overline{2x-x^2}|^{-\frac{3}{2}} + x^{-2}\dot{x} \times \overline{2x^{-1}-1}|^{-\frac{3}{2}} = -\overline{2x-x^2}|^{-\frac{1}{2}} +$

$$\frac{1}{\overline{2x^{-1}-1}|^{\frac{1}{2}}} = \frac{-1}{\sqrt{2x-x^2}} + \frac{x^{\frac{1}{2}}}{\sqrt{2-x}} = -\frac{1}{\sqrt{2x-x^2}} + \frac{x}{\sqrt{2x-x^2}}$$
$$= \frac{x-1}{\sqrt{2x-x^2}}.$$

2. In the same manner, the fluent of $\frac{\dot{x}}{\overline{2x+x^2}|^{\frac{3}{2}}} = -$
$$\frac{1}{\sqrt{2x+x^2}} - \frac{x}{\sqrt{2x+x^2}} = -\frac{x+1}{\sqrt{2x+x^2}}.$$

3. The fluent of $\frac{x\dot{x}}{\sqrt{x-\frac{1}{4}a}} = f.\frac{x^2\dot{x}-\frac{1}{6}ax\dot{x}}{\sqrt{x^3-\frac{1}{4}ax^2}} + f.\frac{\frac{1}{6}a\dot{x}}{\sqrt{x-\frac{1}{4}a}}$
$= \frac{2}{3} \times \sqrt{x^3-\frac{1}{4}ax^2} + \frac{a}{3} \times \sqrt{x-\frac{1}{4}a}.$

**Rule 6.** When the given fluxion is affected by two different surds, and the rational quantity without the vinculum is in a given ratio to the fluxion of the variable quantity under it, it may be useful to substitute for one of the surds.

### (273.) FLUENT 72.

To find the fluent of $\frac{z\dot{z} \times \sqrt{a^2-z^2}}{\sqrt{z^2-b^2}}$.

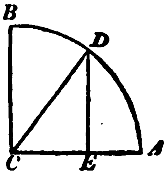
Assume $x = \sqrt{z^2-b^2}$; $\therefore x^2 = z^2-b^2$; and $x\dot{x} = z\dot{z}$. Also $-b^2-x^2 = -z^2$; $\therefore a^2-b^2-x^2 = a^2-z^2$, or if $c^2 = a^2-b^2$, $\sqrt{c^2-x^2} = \sqrt{a^2-z^2}$; therefore the fluxion $= \frac{x\dot{x} \times \sqrt{c^2-x^2}}{x}$ $= \dot{x} \times \sqrt{c^2-x^2} = \frac{c^2\dot{x}-x^2\dot{x}}{\sqrt{c^2-x^2}}$; whose fluent is found in Art. 44. and Fluent 15. The same Rule may be applied in cases like the following.

### (274.) FLUENT 73.

To find the fluent of $\frac{z\dot{z} \times \sqrt{a^2-z^2}}{b^2+z^2}$.

1. Let $b^2+z^2=x^2$; then $z\dot{z}=x\dot{x}$. Also $\sqrt{a^2-z^2}=\sqrt{a^2+b^2-x^2}=\sqrt{c^2-x^2}$, if $c^2=a^2+b^2$; therefore the given fluxion $=\frac{x\dot{x}\times\sqrt{c^2-x^2}}{x^2}=\frac{\dot{x}\times\sqrt{c^2-x^2}}{x}$; whose fluent [illegible] [illegible] $=\frac{c}{2}\times$ hyp. log. $\frac{c-\sqrt{c^2-x^2}}{c+\sqrt{c^2-x^2}}+\sqrt{c^2-x^2}$.

2. The fluent of $\frac{z\dot{z}\times\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}$ is found in the same manner. If $x$ be assumed $=\sqrt{c^2-z^2}$, the fluent will be found $=-$ a circular area $BCED$ whose radius is $\sqrt{b^2+c^2}$, and cosine $x$.



## (275.) FLUENT 74.

Nearly in the same manner we may find the fluent of $\overline{a+cz^n}|^m\times dz^{rn-1}\dot{z}$, where the index of $z$ without the vinculum, increased by unity, is some multiple of the index of $z$ under the vinculum.

Let $a+cz^n=x$; then $z^n=\frac{x-a}{c}$; and $z^{rn}=\frac{\overline{x-a}|^r}{c^r}$; $\therefore rnz^{rn-1}\dot{z}=\frac{r\times\overline{x-a}|^{r-1}\dot{x}}{c^r}$; and $z^{rn-1}\dot{z}=\frac{1}{nc^r}\times\overline{x-a}|^{r-1}\times\dot{x}$; or if $r-1=s$, $dz^{rn-1}\dot{z}=\frac{d}{nc^r}\times\overline{x-a}|^s\times\dot{x}=$ (by the binomial theorem,) $\frac{d\dot{x}}{nc^r}\times\overline{x^s-sax^{s-1}+s.\frac{s-1}{2}a^2x^{s-2}-\&c.}$ Now for $dz^{rn-1}\dot{z}$ substitute this quantity, and $x^m$ for $\overline{a+cz^n}|^m$, and the given fluxion will become $\frac{d}{nc^r}\times x^m\dot{x}\times\overline{x^s-sax^{s-1}+s.\frac{s-1}{2}a^2x^{s-2}-\&c.}$ $=\frac{d}{nc^r}\times\overline{x^{m+s}\dot{x}-sax^{m+s-1}\dot{x}+s.\frac{s-1}{2}a^2x^{m+s-2}\dot{x}-\&c.}$ Hence the fluent $= \frac{d}{nc^r} \times : \frac{x^{m+s+1}}{m+s+1} - \frac{sax^{m+s}}{m+s} + \frac{s \,.\, \overline{\frac{s-1}{2}}\, a^2 x^{m+s-2}}{m+s-1}$ $-$ &c.

On this Fluent it is useful to observe,

1. The series, arising from the expansion of $\overline{x-a}|^r$, will terminate, if $r$, and consequently $s$, be a whole *positive* number; therefore if $m$ be a *positive* whole number, or a *positive* or *negative* fraction, the fluent can be found.

2. If $r$ be a *positive* whole number, and $m$ a *negative* whole number greater than $s+1$, or $r$, the fluent can be found.

3. But if $m$ be a *negative* whole number equal to or less than $r$, the fluxion of one of the terms becomes $\frac{\dot{x}}{x}$, which is found not by the common forms, but by logarithms.

4. By Reduction, the original fluxion becomes $\overline{az^{-n}+c}|^m \times dz^{\overline{m+r}.n-1}\dot{z}$; therefore if $m$ and $r$ be both fractions, but $m+r$ a *negative* whole number, the fluent can be found by transforming the fluxion as before. The series will always terminate, and the fluent of each term may be obtained by the common method.

### (276.) FLUENT 75.

If $a=r^2$, $c=-1$, $n=2$, $m=\frac{1}{2}$, $d=1$, and $r=2$, this fluxion becomes $z^3\dot{z} \times \sqrt{r^2-z^2}$. To find the fluent, let $\sqrt{r^2-z^2}=y$; then $z^2=r^2-y^2$, and $z^4=r^4-2r^2y^2+y^4$; $\therefore z^3\dot{z}=-r^2y\dot{y}+y^3\dot{y}$; and $z^3\dot{z} \times \sqrt{r^2-z^2}=-r^2y^2\dot{y}+y^4\dot{y}$; whose fluent $=-\frac{1}{3}r^2y^3+\frac{1}{5}y^5$. Now when $z=0$, suppose $y=r$; the cor. $=\frac{2r^5}{15}$; and the whole corrected fluent, when $z=r$ (or $y=0$), is $\frac{2r^5}{15}$. This fluent is used in finding the center of gyration of a sphere. For another method, see Fluent 11.

**Rule 7.** A trinomial may be reduced to a binomial, by substituting for the variable quantity and half the coefficient of the middle term with its proper sign.

### (277.) Fluent 76.

Required the fluent of $\frac{\dot z}{\sqrt{b^2+cz+z^2}}$.

Let $x=z+\frac{1}{2}c$; then $x^2=z^2+cz+\frac{1}{4}c^2$; ∴ $x^2+b^2-\frac{c^2}{4}=z^2+cz+b^2$; that is, if $a^2=b^2-\frac{c^2}{4}$, the given fluxion $=\frac{\dot x}{\sqrt{a^2+x^2}}$; and the fluent = hyp. log. $x+\sqrt{a^2+x^2}$.

### (278.) Fluent 77.

Required the fluent of $\frac{z^{n-1}\dot z}{\sqrt{a+bz^n+cz^{2n}}}$.

This $=\frac{z^{n-1}\dot z}{\sqrt{c}\times\sqrt{\frac{a}{c}+\frac{bz^n}{c}+z^{2n}}}$. Assume $z^n+\frac{b}{2c}=x$; then $z^{2n}+\frac{bz^n}{c}+\frac{b^2}{4c^2}=x^2$, and $\sqrt{z^{2n}+\frac{bz^n}{c}+\frac{a}{c}}=\sqrt{x^2+\frac{a}{c}-\frac{b^2}{4c^2}}$ $=\sqrt{x^2\pm d^2}$; if $d^2=\frac{a}{c}-\frac{b^2}{4c^2}$, and $\frac{\dot x}{n}=z^{n-1}\dot z$; therefore the given fluxion $=\frac{\dot x}{n\sqrt{c}\times\sqrt{x^2\pm d^2}}$; whose fluent $=\frac{1}{n\sqrt{c}}\times$ hyp. log. $x+\sqrt{x^2\pm d^2}$.

### (279.) Fluent 78.

Required the fluent of $\dot z\times\sqrt{a+bz+cz^2}$.

Assume $z+\frac{b}{2c}=x$; then by the last case the fluxion may be reduced to the form $\dot{x} \times \sqrt{c} \times \sqrt{x^2 + d^2}$; whose fluent is found in Art. 259. Fluent 58.

### (280.) FLUENT 79.

Required the fluent of $z^{n-1}\dot{z} \times \sqrt{a + bz^n + cz^{2n}}$.

Assume $z^n + \frac{b}{2c} = x$; $\therefore$ $z^{n-1}\dot{z} = \frac{\dot{x}}{n}$; and $z^{2n} + \frac{bz^n}{c} + \frac{b^2}{4c^2} = x^2$; $\therefore$ $\sqrt{z^{2n} + \frac{bz^n}{c} + \frac{a}{c^2}} = \sqrt{x^2 \pm d^2}$, when $\pm d^2 = \frac{a}{c} - \frac{b^2}{4c^2}$; and the given fluxion $= \frac{\dot{x} \times \sqrt{c} \times \sqrt{x^2 \pm d^2}}{n}$, whose fluent is found in Fluent 58.

### (281.) FLUENT 80.

Required the fluent of $\frac{z^{n-1}\dot{z}}{a + bz^n + cz^{2n}}$.

Assume $z^n + \frac{b}{2c} = x$; then the fluxion may be reduced to the form $\frac{\dot{x}}{nc \times \overline{x^2 \pm d^2}}$, and the fluent found as before.

### (282.) FLUENT 81.

To find the fluent of $\frac{cx\dot{x} - d\dot{x}}{x^2 - ax + b}$.

Let $x - \frac{a}{2} = z$; then $\dot{x} = \dot{z}$, and $x^2 - ax + \frac{a^2}{4} = z^2$; $\therefore$ $x^2 - ax + b = z^2 + b - \frac{a^2}{4} = \left(\text{if } e^2 = b - \frac{a^2}{4}\right)$ $z^2 \pm e^2$, according as $e^2$ is positive or negative; that is, according as the two values of $x$ are impossible, or possible. Also $cx\dot{x} - d\dot{x} = cz\dot{z} + \frac{ac\dot{z}}{2} - d\dot{z}$; therefore the given fluxion is transformed into $\frac{cz\dot{z} + \overline{\frac{1}{2}ac - d} \times \dot{z}}{z^2 \pm e^2}$ $=$ (if $\frac{1}{2}ac - d = m$) $\frac{cz\dot{z} + m\dot{z}}{z^2 \pm e^2}$: and the fluent $= \frac{1}{2}c \times$ hyp. log.

$z^2 \pm e^2 + \frac{m}{e^2} \times$ a circular arc, of radius $e$ and tangent $z$, if $e^2$ be positive; and if $e^2$ be negative, the last part $= \frac{m\dot{z}}{z^2 - e^2} = - \frac{m}{2e} \times \frac{2e\dot{z}}{e^2 - z^2}$, and its fluent is $- \frac{m}{2e} \times$ hyp. log. $\frac{e+z}{e-z}$. Let the fluent of this second part $= A$; then the whole required fluent is $\frac{1}{2} c \times$ hyp. log. $\overline{z^2 \pm e^2} + A$.

### (283.) FLUENT 82.

To find the fluent of $\frac{x\dot{x}}{\sqrt{a^2 - ax + x^2}}$.

Assume $x - \frac{a}{2} = y$; then, as before, $\sqrt{x^2 - ax + a^2} = \sqrt{y^2 + \frac{3a^2}{4}}$.

Also $x^2 = y^2 + ay + \frac{a^2}{4}$; $\therefore x\dot{x} = y\dot{y} + \frac{1}{2} a\dot{y}$; and the given fluxion $= \frac{y\dot{y}}{\sqrt{y^2 + \frac{3a^2}{4}}} + \frac{\frac{1}{2} a\dot{y}}{\sqrt{y^2 + \frac{3a^2}{4}}}$; and the fluent $= \sqrt{y^2 + \frac{3a^2}{4}} + \frac{1}{2} a \times$ hyp. log. of $y + \sqrt{y^2 + \frac{3a^2}{4}}$.

### (284.) FLUENT 83.

In the same manner the fluent of $\frac{x^2\dot{x}}{\sqrt{a^2 - ax + x^2}}$ may be resolved into $\frac{y^2\dot{y}}{\sqrt{y^2 + \frac{3a^2}{4}}}$, $\frac{ay\dot{y}}{\sqrt{y^2 + \frac{3a^2}{4}}}$, and $\frac{\frac{a^2}{4}\dot{y}}{\sqrt{y^2 + \frac{3a^2}{4}}}$, of which the first is known by Fluent 14.; the second $= a \times \sqrt{y^2 + \frac{3a^2}{4}}$; and the third $= \frac{a^2}{4} \times$ hyp. log. of $y + \sqrt{y^2 + \frac{3a^2}{4}}$.

RULE 8. If the fluxional quantity without the trinomial be not properly related to the fluxion of the highest power of the variable quantity in it, after substitution the binomial may be expanded, and the fluent of each term taken separately. If the series terminates, the whole fluent is obtained; if not, this is only an approximation to its value.

### (285.) FLUENT 84.

Required the fluent of $\frac{z^m\dot z}{a+bz+cz^2}$.

Let $z+\frac{b}{2c}=x$; then $\dot z=\dot x$, and $z^m=\overline{x-\frac{b}{2c}}\Big|^m$. Also $z^2+\frac{bz}{c}+\frac{a}{c}=x^2\pm d^2$, if $\pm d^2=\frac{a}{c}-\frac{b^2}{4c}$, and the fluxion $=$ $\frac{\dot x\times\overline{x-\frac{b}{2c}}\Big|^m}{x^2\pm d^2}$; where $m$ being by supposition an affirmative integer, $\overline{x-\frac{b}{2c}}\Big|^m$ must be expanded, and the fluent of each term be taken by continuation.

This is of the same form with the 4th of De Moivre's, Coroll. 1.

### (286.) FLUENT 85.

To find the fluent of $\frac{z^{-m}\dot z}{a+bz+cz^2}$.

Let $\frac{1}{y}=z$; $\therefore$ $\dot z=\frac{-\dot y}{y^2}$, and $z^{-m}=y^m$, and $a+bz+cz^2$ $=a+\frac{b}{y}+\frac{c}{y^2}$; therefore the fluxion $=\frac{-y^m\dot y}{ay^2+by+c}$, whose fluent may be found by the last case.

This is of the same form with De Moivre's 7th, Coroll. 1.

(287.) **FLUENT 86.**

Required the fluent of $\frac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}}$.

Let $z^n=y$; then $z^{rn}=y^r$, and $z^{rn-1}\dot{z}=\frac{y^{r-1}\dot{y}}{n}$; the denominator becomes $a+by+cy^2$, and the whole fluxion $=$ $\frac{\frac{1}{n}\times y^{r-1}\dot{y}}{a+by+cy^2}$; in which $r-1$ being assumed an affirmative integer, the fluent can be found by Fluent 84.

(288.) **FLUENT 87.**

To find the fluent of $\frac{z^{-rn-1}\dot{z}}{a+bz^n+cz^{2n}}$.

Let $z^n=\frac{1}{y}$; then $z^{-rn}=y^r$; and $z^{-rn-1}\dot{z}=-\frac{y^{r-1}\dot{y}}{n}$. Also the denominator $=a+\frac{b}{y}+\frac{c}{y^2}$; and the whole fluxion $=-$ $\frac{\frac{1}{n}\times y^{r+1}\dot{y}}{ay^2+by+c}$, whose fluent can be found by Fluent 84.

This and the last are of a form similar to the 12th of De Moivre's.

(289.) **FLUENT 88.**

To find the fluent of $\frac{z^m\dot{z}}{\sqrt{a+bz+cz^2}}$, $m$ being an integer.

Assume $z+\frac{b}{2c}=x$; then $\dot{z}=\dot{x}$, and $z^m=\overline{x-\frac{b}{2c}}\big|^m$; and the whole may be reduced to the form $\frac{\overline{x-\frac{b}{2c}}\big|^m\times\dot{x}}{\sqrt{c}\times\sqrt{x^2\pm d^2}}$; whose fluent may be found by expanding the binomial.

### (290.) FLUENT 89.

1. To find the fluent of $\frac{z^{-m}\dot{z}}{\sqrt{a+bz+cz^2}}$.

Let $z=\frac{1}{y}$, then $z^{-m}=y^m$, and $\dot{z}=\frac{-\dot{y}}{y^2}$; hence the fluxion becomes, by the same methods, $=\frac{-y^{m-1}\dot{y}}{\sqrt{ay^2+by+c}}$, whose fluent may be found by the last.

2. The fluent of $\frac{z^{m-1}\dot{z}}{\sqrt{a+bz^n+cz^n}}$ may be found in the same manner, by assuming $y=z^n$.

3. And that of $\frac{z^{-mn-1}\dot{z}}{\sqrt{a+bz^n+cz^n}}$, by taking $z^n=\frac{1}{y}$.

### (291.) FLUENT 90.

1. For the fluent of $z^m\dot{z}\times\sqrt{a+bz+cz^2}$, take $z+\frac{b}{2c}=x$; it may be thus reduced to the form $\overline{x-\frac{b}{2c}}\Big|^{m}\times\dot{x}\times\sqrt{c}\times\sqrt{x^2+d^2}$, the fluent of which is found by expanding $\overline{x-\frac{b}{2c}}\Big|^{m}$.

2. For the fluent of $z^{-m}\dot{z}\times\sqrt{a+bz+cz^2}$, assume $z=\frac{1}{y}$ the given fluxion $=-\dot{y}y^{m-3}\times\sqrt{ay^2+by+c}$, whose fluent is found by the last case.

3. For that of $z^{m-1}\dot{z}\times\sqrt{a+bz^n+cz^{2n}}$ take $z^n$ also $=y$.

4. And for that of $z^{-mn-1}\dot{z}\ \sqrt{a+bz^n+cz^{2n}}$, let $z^n=\frac{1}{y}$.

### (292.) FLUENT 91.

To find the fluent of $\frac{x\dot{x}}{\sqrt{2x-4ax+2ax^2}}$.

This $= x\dot{x} \times \overline{2x - 4nx + 2nx^2}\big|^{-\frac{1}{2}}$. Assume $2x = a$, $4nx - 2nx^2 = b$; then since $\overline{a - b}\big|^m = a^m - ma^{m-1}b + m.\frac{m-1}{2}a^{m-2}b^2 - \&c.$ by substituting for $a$, $b$ and $m$ their values, $\overline{2x - 4nx + 2nx^2}\big|^{-\frac{1}{2}}$ $= \frac{x^{-\frac{1}{2}}}{2^{\frac{1}{2}}} + \frac{nx^{-\frac{1}{2}}}{2^{\frac{3}{2}}} - \frac{nx^{\frac{1}{2}}}{2^{\frac{3}{2}}} + \&c.$; and the fluxion $= \frac{x^{\frac{1}{2}}\dot{x}}{2^{\frac{1}{2}}} + \frac{nx^{\frac{1}{2}}\dot{x}}{2^{\frac{1}{2}}} - \frac{nx^{\frac{3}{2}}\dot{x}}{2^{\frac{3}{2}}} + \&c.$; whose fluent $= \frac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3} + \frac{2^{\frac{1}{2}}nx^{\frac{3}{2}}}{3} - \frac{nx^{\frac{5}{2}}}{5 \times 2^{\frac{1}{2}}} + \&c.$

This fluent is used in Art. [illegible] 135. Prob. 9. in the Attraction of Bodies.

### (293.) FLUENT 92.

To find the fluent of $\frac{x\dot{x}}{\sqrt{2x + 4nx - 2nx^2}}$.

Assume $a = 2x$; $b = 4nx - 2nx^2$; then since $\overline{a + b}\big|^m = a^m + ma^{m-1}b + m.\frac{m-1}{2}a^{m-2}b^2 + \&c.$ by substituting as in the last case, we have $x\dot{x} \times \overline{2x + 4nx - 2nx^2}\big|^{-\frac{1}{2}} = \frac{x^{\frac{1}{2}}\dot{x}}{2^{\frac{1}{2}}} + \frac{nx^{\frac{3}{2}}\dot{x}}{2^{\frac{3}{2}}} - \frac{nx^{\frac{1}{2}}\dot{x}}{2^{\frac{1}{2}}} + \&c$; and the fluent $= \frac{2^{\frac{1}{2}}x^{\frac{3}{2}}}{3} + \frac{nx^{\frac{5}{2}}}{5 \times 2^{\frac{5}{2}}} - \frac{2^{\frac{1}{2}}nx^{\frac{3}{2}}}{3} + \&c.$

This fluent is used in Art. [illegible] 136. Prob. 10. on the Attraction of Bodies. In both these cases, if $n$ be very small, the succeeding terms vanish, and the whole fluent is expressed by the three first terms of the series.

### (294.) FLUENT 93.

To find the fluent of $\frac{a\dot{y}}{\sqrt{a^2 - y^2}}$ in a series.

In cases of this kind, no substitution is necessary.

For $a^2 - y^2 = a^2 \times \overline{1 - \frac{y^2}{a^2}}$; $\therefore \frac{a\dot{y}}{\sqrt{a^2 - y^2}} = \frac{\dot{y}}{\sqrt{1 - \frac{y^2}{a^2}}} = \dot{y} \times$ $\overline{1 - \frac{y^2}{a^2}}\Big|^{-\frac{1}{2}}$; which, expanded by the series $\overline{a - b}\big|^{m}$, $= \dot{y} \times : 1 + \frac{y^2}{2a^2} + \frac{3y^4}{4a^4} +$ &c. $= \dot{y} + \frac{y^2\dot{y}}{2a^2} + \frac{3y^4\dot{y}}{4a^4}$ + &c.; and the fluent $= y + \frac{y^3}{2 . 3 . a^2} + \frac{3y^5}{4 . 5 . a^4}$ + &c.

This fluent is used in Art. 55. Ex. 5.

(295.) Rule 9. When the denominator is a rational trinomial or multinomial, the best way of proceeding in general is to resolve the given fraction into binomial ones, by assuming the denominator = 0, and finding its roots.

### (296.) Fluent 94.

To find the fluent of $\frac{\dot{x}}{x^2 + ax + b}$.

Let $x^2 + ax + b = 0$; then the roots are $-\frac{1}{2}a - \sqrt{\frac{a^2}{4} - b}$, and $-\frac{a}{2} + \sqrt{\frac{a^2}{4} - b}$. Let these quantities $= p$ and $q$; then $\overline{x - p} \times \overline{x - q} = x^2 + ax + b$.

Let then $\frac{A}{x - p} + \frac{B}{x - q} = \frac{1}{x^2 + ax + b}$; then $\frac{A \times \overline{x - q} + B \times \overline{x - p}}{\overline{x - p} \times \overline{x - q}} = \frac{1}{x^2 + ax + b}$; hence, the numerators being equal as well as the denominators, we have

$$\left.\begin{array}{r} Ax - Aq \\ + Bx - Bp \\ - 1 \end{array}\right\} = 0;$$

$\therefore Ax + Bx = 0$, and $A = -B$; also $-Aq - Bp = 1$; or, since $A = -B$, $Ap - Aq = 1$; $\therefore A = \frac{1}{p - q}$, and $B = \frac{1}{q - p}$;

∴ since $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}=\frac{\dot{x}}{x^2+ax+b}$, we have $\frac{\dot{x}}{\overline{p-q}\times\overline{x-p}}+\frac{\dot{x}}{\overline{q-p}\times\overline{x-q}}=\frac{\dot{x}}{x^2+ax+b}$; and the fluent $=\frac{1}{p-q}\times$ hyp. log. $\overline{x-p}+\frac{1}{q-p}\times$ hyp. log. $\overline{x-q}$.

(297.) The quantities $A$, $B$, &c. may be determined more easily, by substituting one of its roots for $x$ after the multiplication, in which case most of the terms will vanish.

Thus, let $p$, $q$, $r$, &c. be roots of the equation $x^n-Px^{n-1}$ + &c., to resolve the fraction $\frac{1}{x^n-Px^{n-1}+\&c.}$ into $\frac{K}{x-p}+\frac{L}{x-q}+\frac{M}{x-r}+$ &c.

Reduce the fractions to a common denominator; then the denominators on both sides of this equation will be equal, and the sum of the numerators will $=1$. That is, $K\times\overline{x-q}\times\overline{x-r}\times$ &c. $+L\times\overline{x-p}\times\overline{x-r}\times$ &c. $+M\times\overline{x-p}\times\overline{x-q}\times$ &c. $=1$. Now as this is true for every value of $x$, let $x=p$; then $x-p=0$; ∴ $K\times\overline{p-q}\times\overline{p-r}\times$ &c. $=1$, and $K=\frac{1}{\overline{p-q}\times\overline{p-r}\times\&c.}$. Next let $x=q$; then $x-q=0$, and $L\times\overline{q-p}\times\overline{q-r}\times$ &c. $=1$; ∴ $L=\frac{1}{\overline{q-p}\times\overline{q-r}\times\&c.}$; and in the same manner the other numerators may be determined.

### (298.) FLUENT 95.

To find the fluent of $\frac{x^2\dot{x}}{x^3+ax^2+bx+c}$.

Assume $p$, $q$, and $r$, the roots; and let $\frac{K}{x-p}+\frac{L}{x-q}+\frac{M}{x-r}=\frac{1}{x^3+ax^2+bx+c}$; then $K=\frac{1}{\overline{p-q}\times\overline{p-r}}$; $L=$

$\frac{1}{\overline{q-p}\times\overline{q-r}}$; $M=\frac{1}{\overline{r-p}\times\overline{r-q}}$; and the given fluxion $=$ $\frac{Kx^2\dot{x}}{x-p}+\frac{Lx^2\dot{x}}{x-q}+\frac{Mx^2\dot{x}}{x-r}$; of which the fluents, by actual division, are $K\times\frac{x^2}{2}+Kpx+Kp^2\times$ hyp. log. $\overline{x-p}+L\times\frac{x^2}{2}+Lqx+Lq^2\times$ hyp. log. $\overline{x-q}+M\times\frac{x^2}{2}+Mrx+Mr^2\times$ hyp. log. $\overline{x-r}=\overline{K+L+M}\times\frac{x^2}{2}+\overline{Kp+Lq+Mr}\times x+Kp^2\times$ hyp. log. $\overline{x-p}+Lq^2\times$ hyp. log. $\overline{x-q}+Mr^2\times$ hyp. log. $\overline{x-r}$.

## (299.) FLUENT 96.

To find the fluent of $\frac{x^m\dot{x}}{x^n-Px^{n-1}+\&c.}$, $m$ being a whole positive number, and the roots of the denominator $p$, $q$, $r$, &c.

This fluxion may be resolved, like the last, into the following quantities; $\frac{Kx^m\dot{x}}{x-p}+\frac{Lx^m\dot{x}}{x-q}+$ &c.; where $K$, $L$, &c. may be determined as before, and the fluents may be found in each case by actual division. The first fluent $=\frac{Kx^m}{m}+\frac{Kpx^{m-1}}{m-1}$ $+$ &c. $+\,Kp^m\times$ hyp. log. $\overline{x-p}$; in like manner the fluents of the other quantities are found. The sum of all these is the fluent required; and it $=\overline{K+L+\&c.}\times\frac{x^m}{m}+$ $\overline{Kp+Lq+\&c.}\times\frac{x^{m-1}}{m-1}+$ &c. $+\,Kp^m\times$ hyp. log. $\overline{x-p}+$ $Lq^m\times$ hyp. log. $\overline{x-q}+$ &c.

To determine when any of these coefficients become equal to nothing, Vide Dr. Waring's *Medit. Algeb.* in the Addenda.

(300.) If two roots, as $p$ and $q$, are equal, one of the quantities must have a quadratic divisor $\overline{x-p}|^2$.

Let $\dfrac{1}{x^3-ax^2+bx-c}=\dfrac{Lx+M}{\overline{x-p}|^2}+\dfrac{N}{x-r}$; reduce them to a common denominator, and equate the numerators;

$$\text{then}\ \left.\begin{array}{lll} Lx^2 & -\ Lrx & -\ Mr \\ +\ Nx^2 & +\ Mx & +\ Np^2 \\ & -\ 2Npx & -\ 1 \end{array}\right\}=0;$$

hence, $L+N=0$; $M-Lr-2Np=0$; $-Mr+Np^2-1=0$;

$\therefore\ L=-N$; $M=\dfrac{Np^2-1}{r}$; $\therefore\ \dfrac{Np^2-1}{r}+Nr-2Np=0$;

$\therefore\ N=\dfrac{1}{\overline{p-r}|^2}$; $L=\dfrac{1}{\overline{-r-p}|^2}$; $M=\dfrac{2p-r}{\overline{p-r}|^2}$. Hence the fluent of $\dfrac{\dot x}{x^3-ax^2+bx-c}=\int.\ \dfrac{Lx\dot x+M\dot x}{\overline{x-p}|^2}+\dfrac{N\dot x}{x-r}$; where $L$, $M$, and $N$ are found. And this fluent $=L\times$ hyp. log. $\overline{x-p}-\dfrac{Lp+M}{x-p}+N\times$ hyp. log. $\overline{x-r}$. (Fluent 54.)

(301.) If some of the roots $p$, $q$, $r$, &c. be impossible, the fractions, in which the impossible roots are contained, must be incorporated in pairs; and then the impossible quantities will disappear. For if $\pm a+\sqrt{-b^2}$, and $\pm a-\sqrt{-b^2}$ be the roots of an equation of two dimensions, since they enter by pairs, the equation itself is $\overline{x\mp a-\sqrt{-b^2}}\times\overline{x\mp a+\sqrt{-b^2}}$, or $x^2\mp 2ax+a^2+b^2=0$; where the imaginary quantities are not found.

## (302.) FLUENT 97.

Let the proposed fraction be $\dfrac{x\dot x}{x^3+ax^2+bx+c}$, and let two roots, $p$ and $q$, of the cubic equation be impossible.

Assume $\dfrac{L}{x-p}+\dfrac{M}{x-q}+\dfrac{N}{x-r}=\dfrac{1}{x^3+ax^2+bx+c}$, and incorporate the two first; then we have the given fluxion $=$

$\dfrac{\overline{L+M} \times x\dot{x} - \overline{Lq+Mp} \times \dot{x}}{x^2 - \overline{p+q} \times x + pq} + \dfrac{N\dot{x}}{x-r}$; where the impossible parts vanish, as will appear by substituting $a + \sqrt{-b^2}$ and $a - \sqrt{-b^2}$ for $p$ and $q$. The fluent of the first part is found by the method of Fluent 94.; and the fluent of the last is $N \times$ hyp. log. $\overline{x-r}$.

## (303.) FLUENT 98.

In general, to find the fluent of $\dfrac{x^m \dot{x}}{x^2 - ax + b}$.

If the roots be both possible, resolve $\dfrac{1}{x^2 - ax + b}$ into $\dfrac{K}{x-p} + \dfrac{L}{x-q}$; and the given fluxion $= \dfrac{Kx^m\dot{x}}{x-p} + \dfrac{Lx^m\dot{x}}{x-q}$, whose fluents have already been found.

But if the roots be impossible, divide $x^m\dot{x}$ by $x^2 - ax + b$, until the remainder becomes $cx\dot{x} - d\dot{x}$, $c$ and $d$ being the coefficients, which arise from the division; let the quotient be $x^{m-2}\dot{x} + ax^{m-3}\dot{x} + \overline{a^2 - b} \,.\, x^{m-4}\dot{x} + \&c. + \dfrac{cx\dot{x} - d\dot{x}}{x^2 - ax + b}$; and

$$\text{the fluent} = \frac{x^{m-1}}{m-1} + \frac{ax^{m-2}}{m-2} + \frac{\overline{a^2-b} \,.\, x^{m-3}}{m-3} + \&c. + f.\, \frac{cx\dot{x} - d\dot{x}}{x^2 - ax + b},$$

or $+ \frac{1}{2} c \times$ hyp. log. $\overline{z^2 \pm e^2} + A$. (Fluent 81.)

(304.) RULE 10. When the denominator of the fluxion is of the form $\overline{x+a}|^{m} \times \overline{x+b}|^{n}$, the given fraction must be resolved into others of the form $\dfrac{L}{\overline{x+a}|^{m}} + \dfrac{M}{\overline{x+a}|^{m-1}} + \dfrac{N}{\overline{x+a}|^{m-2}} + \&c.$ $+ \dfrac{P}{\overline{x+b}|^{n}} + \dfrac{Q}{\overline{x+b}|^{n-1}} + \dfrac{R}{\overline{x+b}|^{n-2}} + \&c.$, respectively continued to $m$ and $n$ quantities.

(305.) LEMMA. To resolve $\dfrac{1}{\overline{x+a}|^{m} \times \overline{x+b}|^{n}}$ into $\dfrac{L}{\overline{x+a}|^{m}}$ $+ \dfrac{M}{\overline{x+a}|^{m-1}} + \&c.$

Reduce the fractions to a common denominator, and suppose the numerators on each side equal; then $L\times\overline{x+b}|^{n}+M\times\overline{x+b}|^{n}\times\overline{x+a}+N\times\overline{x+b}|^{n}\times\overline{x+a}|^{2}$ + &c. + $P\times\overline{x+a}|^{m}+Q\times\overline{x+a}|^{m}\times\overline{x+b}+R\times\overline{x+a}|^{m}\times\overline{x+b}|^{2}$ + &c. $=1$. Call this equation $A$. Now assume $x+a=0$; then $x=-a$, and all those terms, into which $x+a$ enters, $=0$; $\therefore L\times\overline{b-a}|^{n}=1$, and $L=\frac{1}{\overline{b-a}|^{n}}$. Take the fluxion of the equation $A$, and divide by $\dot{x}$; then $nL\times\overline{x+b}|^{n-1}+nM\times\overline{x+b}|^{n-1}\times\overline{x+a}+M\times\overline{x+b}|^{n}$ + &c. $=0$. Let $x=-a$, and we have $nL\times\overline{b-a}|^{n-1}+M\times\overline{b-a}|^{n}=0$; $\therefore M=\frac{-nL}{b-a}=\frac{-n}{\overline{b-a}|^{n+1}}$. Continue to take the fluxion of the last equation, making $x+a=0$, and the values of $N$, &c. will be determined.

Again, assume $x+b=0$, and $P$ will be found $=\frac{1}{\overline{a-b}|^{m}}$; take the fluxion, and again assume $x+b=0$; then $Q=\frac{-m}{\overline{a-b}|^{m+1}}$, and so on for $R$, $S$, &c.

## (306.) FLUENT 99.

To find the fluent of $\frac{x^{r}\dot{x}}{\overline{x+a}|^{m}\times\overline{x+b}|^{n}}$.

By the last Article, the given fluxion $=\frac{Lx^{r}\dot{x}}{\overline{x+a}|^{m}}+\frac{Mx^{r}\dot{x}}{\overline{x+a}|^{m-1}}$ + &c. + $\frac{Px^{r}\dot{x}}{\overline{x+b}|^{n}}+\frac{Qx^{r}x}{\overline{x+b}|^{n-1}}$ + &c. Assume $x+a=z$; then $x^{r+1}=\overline{z-a}|^{r+1}$, and $x^{r}\dot{x}=\dot{z}\times\overline{z-a}|^{r}$; $\therefore \frac{Lx^{r}\dot{x}}{\overline{x+a}|^{m}}=L\times\frac{\overline{z-a}|^{r}\dot{z}}{z^{m}}$ = by actual expansion, $L\times : z^{r-m}\dot{z}-raz^{r-m-1}\dot{z}+r.\overline{\frac{r-1}{2}}\times a^{2}z^{r-m-2}\dot{z}$ − &c. to $r+1$ terms. The fluent of every term in this series can be found by the common method, except that where the index of $z$ is $-1$, which involves the logarithm of $z$. In the same manner the fluents of the other series can be determined, and their sum is the fluent required.

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## SECTION V.

(307.) The sixteen following Fluents are taken from De Moivre's *Miscellanea Analytica,* lib. iii.

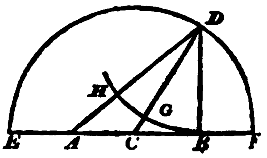
### Fluent 100. (*De Moivre* 1.)

To find the fluent of $\frac{\dot z}{1+2az+z^2}$ $(\dot A)$, where $a$ is less than 1, or the roots of the denominator impossible.

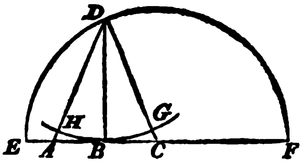
Let $v=z+a$; then $1+2az+z^2=v^2-a^2+1=s^2+v^2$, if $1-a^2=s^2$; and $\dot z=\dot v$; ∴ the given fluxion $=\frac{\dot v}{s^2+v^2}=\frac{1}{s}\times\frac{s^2\dot v}{s^2+v^2}$; and the fluent $=\frac{1}{s^2}\times$ a circular arc of radius $s$, and tangent $v$; or of radius $=\sqrt{1-a^2}$, and tangent $z+a$.

Hence this construction:

With $C$ as center, and $CE=1$ as radius, describe the semicircle $EDF$. Take $CA=z$, and $CB$ on the other side of $C=a$; draw the ordinate $BD$; with $D$ as center, and $DB$ radius, describe a circular arc $BGH$; then $BA=z+a=v$; and $DB=s=\sqrt{1-a^2}$; ∴ the fluent $=\frac{1}{DB^2}\times$ the arc $BH$.



(308.) The fluent of $\frac{\dot z}{1-2az+z^2}$ is found by taking $z-a=v$; and it $=\frac{1}{s^2}\times$ a circular arc of radius $s$, and tangent $z-a$.



Here take $CA = z$, $CB = a$ on the same side of $C$, and draw the ordinate $BD$; join $DC$, $DA$; then the fluent $= \frac{1}{DB^2} \times$ the arc $BH$.

(309.) The fluent of $\frac{\dot{z}}{r^2 + 2arz + z^2}$, where $a$ is less than 1, may be found in the same manner by assuming $\frac{z}{r} + a = v$; for in this case $r^2 + 2arz + z^2 = r^2 \times \overline{s^2 + v^2}$; and $\dot{z} = r\dot{v}$; ∴ the given fluxion $= \frac{r\dot{v}}{r^2 \times \overline{s^2 + v^2}} = \frac{1}{rs^2} \times \frac{s^2\dot{v}}{s^2 + v^2}$; and the fluent $= \frac{1}{rs^2} \times$ a circular arc of radius $s = \sqrt{1 - a^2}$, and tangent $\frac{z}{r} + a$.

(310.) And the fluent of $\frac{\dot{z}}{r^2 - 2arz + z^2}$, where $a$ is less than 1, may be found by taking $\frac{z}{r} - a = v$.

### FLUENT 101. (*De Moivre* 2.)

(311.) To find the fluent of $\frac{z\dot{z}}{1 + 2az + z^2}$ $(\dot{B})$, where $a$ is less than 1.

Let $v = z + a$; then the denominator $= s^2 + v^2$; also $z^2 = v^2 - 2av + a^2$; ∴ $z\dot{z} = v\dot{v} - a\dot{v}$; and the given fluxion $= \frac{v\dot{v} - a\dot{v}}{s^2 + v^2}$; and the fluent $= \frac{1}{2}$ hyp. log. $\overline{s^2 + v^2} - \frac{a}{s^2} \times$ a circular arc of radius $s$, and tangent $v =$ hyp. log. $\sqrt{s^2 + v^2} - \frac{a}{s^2} \times$ arc $BH$ (Fig. to Fluent 100.) $=$ hyp. log. of $AD - \frac{CB}{DB^2} \times$ arc $BH$.

(312.) The fluent of $\frac{z\dot{z}}{1 - 2az + z^2}$ is found by assuming $z - a = v$.

(313.) The fluent of $\frac{z\dot{z}}{r^2+2arz+z^2}$, where $a$ is less than 1, may be found by assuming $\frac{z}{r}+a=v$;

And of $\frac{z\dot{z}}{r^2-2arz+z^2}$, by taking $\frac{z}{r}-a=v$.

### FLUENT 102. (*De Moivre* 3.)

(314.) To find the fluent of $\frac{z^2\dot{z}}{1+2az+z^2}$ ($\dot{C}$), where $a$ is less than 1.

Divide the numerator by the denominator in an inverted order.

$$\begin{array}{r} z^2+2az+1)\,z^2\dot{z}\,(\dot{z} \\ \underline{z^2\dot{z}+2az\dot{z}+\dot{z}} \\ -2az\dot{z}-\dot{z} \end{array}$$

$\therefore\ \dot{C}=\dot{z}-\frac{2az\dot{z}}{1+2az+z^2}-\frac{\dot{z}}{1+2az+z^2}$; and the fluent or $C=z-2aB-A$.

### FLUENT 103. (*De Moivre* 4.)

(315.) To find the fluent of $\frac{z^3\dot{z}}{1+2az+z^2}$ ($\dot{D}$), $a$ being less than 1.

Divide, as before, in an inverted order; then the given fluxion $= z\dot{z}-\frac{2az^2\dot{z}}{1+2az+z^2}-\frac{z\dot{z}}{1+2az+z^2}$;

$$\text{or } \dot{D}=z\dot{z}-2a\dot{C}-\dot{B};\ \therefore$$

$$D=\frac{z^2}{z}-2aC-B.$$

In the same manner the series might be continued to any power of $z$.

(316.) COR. 1. Hence, if $a$ be less than 1, and $m$ any whole positive number, we can determine the fluent of $\frac{z^m\dot{z}}{1+2az+z^2}$.

Divide the numerator by the denominator in an inverted order, till the remainder becomes $\pm\frac{cz\dot{z} \pm e\dot{z}}{1 + 2az + z^2}$, $c$ and $e$ being the coefficients; then the given fluxion $= z^{m-2}\dot{z} - 2az^{m-3}\dot{z} + 2bz^{m-4}\dot{z} -$ &c. $\pm \frac{cz\dot{z} \pm e\dot{z}}{1 + 2az + z^2}$; and the required fluent $= \frac{z^{m-3}}{m-3} - \frac{2az^{m-4}}{m-4} +$ &c. $\pm cB \pm eA$.

(317.) Cor. 2. If $a$ be less than 1, and $m$ be a positive fraction, the fluent of $\frac{z^m\dot{z}}{1+2az+z^2}$ can be found.

Let $m = \frac{d}{t}$, a fraction in its lowest terms; then, if $v = z^{\frac{1}{t}}$, $v^d = z^{\frac{d}{t}}$; also $z = v^t$; $\therefore$ $\dot{z} = tv^{t-1}\dot{v}$, and $z^2 = v^{2t}$; hence the given fluxion $= \frac{tv^{d+t-1}\dot{v}}{1+2av^t+v^{2t}}$. But (by Prob. 77.) the fraction $\frac{1}{1+2av^t+v^{2t}}$ is divisible into component parts of this form $\frac{p-qv}{1-2av+v^2}$; therefore this fluxion is reducible to the form $\frac{ptv^{d+t-1}\dot{v}}{1-2bv+v^2} - \frac{qtv^{d+t}\dot{v}}{1-2bv+v^2}$, of which the fluents are known by the last Corollary; for here $b$ is less than 1, and $d$ and $t$ are whole numbers.

### Fluent 104. (*De Moivre* 5.)

(318.) To find the fluent of $\frac{z^{-1}\dot{z}}{1+2az+z^2}$ $(\dot{E})$, $a$ being less than unity.

Divide the numerator by the denominator in the natural order; then $\dot{E} = \frac{\dot{z}}{z} - \frac{2a\dot{z}}{1+2az+z^2} - \frac{z\dot{z}}{1+2az+z^2} = \frac{\dot{z}}{\dot{z}} - 2a\dot{A} - \dot{B}$; and the fluent = hyp. log. of $z - 2aA - B$.

### FLUENT 105. (*De Moivre* 6.)

(319.) To find the fluent of $\frac{z^{-2}\dot{z}}{1+2az+z^2}$ $(\dot{F})$, $a$ being less than unity.

Divide as before; then $\dot{F}=\frac{\dot{z}}{z^2}-2a\dot{E}-\dot{A}$; and the fluent $=-\frac{1}{z}-2aE-A$.

### FLUENT 106. (*De Moivre* 7.)

(320.) To find the fluent of $\frac{z^{-3}\dot{z}}{1+2az+z^2}$ $(\dot{G})$, $a$ being less than unity.

By division, this fluxion $=z^{-3}\dot{z}-2a\dot{F}-\dot{E}$; $\therefore$ the fluent $G=-\frac{1}{2z^2}-2aF-E$; and the fluents may be continued to any power.

(321.) COR. 1. If $m$ be any whole number, the fluent of $\frac{z^{-m}\dot{z}}{1+2az+z^2}$ can be found by the same method.

Or if $v=\frac{1}{z}$, it can be transformed into the fluxion $-\frac{v^m\dot{v}}{1+2av+v^2}$, of which the fluent is known by Art. 316.

(322.) COR. 2. And if $m$ be a fraction, and $a$ less than 1, the fluent can be found by the method adopted in Art. 317.

### FLUENT 107. (*De Moivre* 8.)

(323.) The fluent of $\frac{\dot{z}}{1+2az+z^2}$, where $a$ is greater than 1, is of the same form with Fluent 94, and found in the same manner.

### FLUENT 108. (*De Moivre* 9.)

(324.) To find the fluent of $\frac{z^n\dot{z}}{1+mz}$, $n$ being a whole positive number.

Divide the numerator by the denominator in an inverted order; then the fluent $= \frac{z^n}{mn} - \frac{z^{n-1}}{m^2 \times \overline{n-1}} + \frac{z^{n-2}}{m^3 \times \overline{n-2}} -$ &c. $(n) \pm \frac{1}{m^{n+1}} \times$ hyp. log. $\overline{1+mz}$.

This is similar to Fluent 4.

### FLUENT 109. (*De Moivre* 10.)

(325.) The fluent of $\frac{z^{-n}\dot{z}}{1+mz}$ is found by dividing in the natural order; and it $= -\frac{1}{\overline{n-1}\,.\,z^{n-1}} + \frac{m}{\overline{n-2}\,.\,z^{n-2}} -$ &c. $(n)$ $\pm\, m^{n-1} \times$ hyp. log. $\overline{1+mz}$.

### FLUENT 110. (*De Moivre* 11.)

(326.) To find the fluent of $\frac{z^{\frac{d}{t}}\dot{z}}{1+mz}$.

Take $y = \overline{mz}|^{\frac{1}{t}}$; $\therefore\ mz = y^t$; also $z^{\frac{1}{t}} = \frac{y}{m^{\frac{1}{t}}}$, and $z^{\frac{d+t}{t}} = \frac{y^{d+t}}{m^{\frac{d+t}{t}}}$, $\therefore\ \frac{d+t}{t} \times z^{\frac{d}{t}}\dot{z} = \frac{\overline{d+t} \times y^{d+t-1}\dot{y}}{m^{\frac{d+t}{t}}}$, and $z^{\frac{d}{t}}\dot{z} = \frac{t}{[illegible]} \times \frac{y^{d+t-1}\dot{y}}{m^{\frac{d+t}{t}}}$; but $1+mz = 1+y^t$; $\therefore$ the given fluxion is equal to $\frac{ty^n\dot{y}}{m^{\frac{n+1}{t}} \times \overline{1+y^t}}$, $n$ being substituted for $d+t-1$. Now by Prob. 80. and 81. $\frac{1}{1+y^t}$ can be resolved into quadratic divisors of the form $\frac{q-ry}{1-2ay+y^2}$, &c., where $t$ is even, and $a$ is less than 1;

therefore the resulting fluxions in this case will be of the form $\frac{t}{m^{\frac{n+1}{t}}} \times \frac{qy^n\dot{y} - ry^{n+1}\dot{y}}{1 - 2ay + y^2}$, whose fluents can be found by the preceding Cases.

If $t$ is odd, the first divisor is of the form $\frac{p}{1 \pm y}$, and the first resulting fluxion is $\frac{t}{m^{\frac{n+1}{t}}} \times \frac{py^n\dot{y}}{1+y}$, whose fluent can be found.

(327.) In the same manner the fluent of $\frac{z^{\frac{d}{t}}\dot{z}}{1 - mz}$ can be found by the resolutions in Prob. 78. and 79.

### FLUENT 111. (*De Moivre* 12.)

(328.) To find the fluent of $\frac{z^m\dot{z}}{1 + 2lz^n + z^{2n}}$, where $l$ is less than unity.

By Prob. 77. $\frac{1}{1 + 2lz^n + z^{2n}}$ can be resolved into quadratic divisors of the form $\frac{p - qz}{1 - 2az + z^2}$, where $a$ is less than 1; ∴ by multiplying $z^m\dot{z}$ into these divisors, the resulting fluxions will be of the form $\frac{pz^m\dot{z} - qz^{m+1}\dot{z}}{1 - 2az + z^2}$, of which the fluents are known by the preceding Cases, whatever be the value of $m$.

(329.) If the given fluxion be $\frac{z^m\dot{z}}{1 + 2lz^{-n} + z^{-2n}}$, multiply both the numerator and the denominator by $z^{2n}$, and the fluxion will become $\frac{z^{m+2n}\dot{z}}{z^{2n} + 2lz^n + 1}$, whose fluent can be found.

### FLUENT 112. (*De Moivre* 13.)

(330.) To find the fluent of $\frac{z^{\frac{d}{t}}\dot{z}}{1 \pm 2lz^{\frac{n}{p}} + z^{\frac{2n}{p}}}$.

Take $z^{\frac{1}{tp}} = y$; $\therefore$ $z^{\frac{1}{t}} = y^{p}$, and $z^{\frac{d+t}{t}} = y^{\overline{d+t}\times p}$; hence $\frac{d+t}{t} \times z^{\frac{d}{t}}\dot{z} = \overline{dp+tp}\,y^{dp+tp-1}\dot{y}$, and $z^{\frac{d}{t}}\dot{z} = tp \times y^{dp+tp-1}\dot{y}$. Again, $z^{\frac{1}{p}} = y^{t}$; $\therefore$ $z^{\frac{n}{p}} = y^{tn}$; hence the given fluxion becomes $= \frac{tpy^{dp+tp-1}\dot{y}}{1 \pm 2ly^{tn} \pm y^{2tn}}$, whose fluent is known by the preceding Cases.

### FLUENT 113. (*De Moivre* 14.)

(331.) To find the fluent of $\frac{z^{m}\dot{z}}{1 \pm z^{n}}$.

By Prob. 78, 79, 80, and 81, the quantity $\frac{1}{1 \pm z^{n}}$ can be resolved into divisors, either all trinomials of the form $\frac{q - rz}{1 - 2az + z^{2}}$, where $a$ is less than unity; or into divisors partly trinomial and partly binomial, of the form $\frac{p}{1 - z^{2}}$, or $\frac{p}{1 - z}$; $z^{m}\dot{z}$ is to be multiplied into these divisors, and the fluents can be found as before.

(332.) The fluent of $\frac{z^{m}\dot{z}}{r^{n} \pm z^{n}}$ can be found in the same manner by the resolution of $\frac{1}{r^{n} \pm z^{n}}$.

### FLUENT 114. (*De Moivre* 15.)

(333.) To find the fluent of $\frac{z^{m}\dot{z}}{1 + 2lz^{n} + z^{2n}}$, where $l$ is greater than 1.

Let $z^{n} = x$; then $z^{2n} = x^{2}$; also $z = x^{\frac{1}{n}}$; $\therefore$ $z^{m+1} = x^{\frac{m+1}{n}}$, and $z^{m}\dot{z} = \frac{x^{\frac{m-n+1}{n}}\dot{x}}{n}$; $\therefore$ the given fluxion $= \frac{\frac{1}{n} \times x^{r}\dot{x}}{1 + 2lx + x^{2}}$, where $r = \frac{m-n+1}{n}$; let $\frac{1}{1+2lx+x^2}$ be resolved, as in Fluent 94, into $\frac{A}{x-p}$ and $\frac{B}{x-q}$; then the original fluxion is transformed into two others, whose fluents can be found.

### FLUENT 115. (*De Moivre* 16.)

(334.) To find the fluent of $\frac{z^m\dot{z}}{1\pm 2lz^n+z^{2n}}$, where $l=1$; or given the fluent of $\frac{z^m\dot{z}}{1\pm z^n} = p$, to find the fluent of $\frac{z^m\dot{z}}{\overline{1\pm z^n}\rceil^2}$.

Divide the numerator and denominator by $z^{2n}$; then the fluxion $= \frac{z^{m-2n}\dot{z}}{\overline{z^{-n}\pm 1}\rceil^2} = \frac{z^{-n-1}}{\overline{z^{-n}\pm 1}\rceil^2} \times z^{m-n+1}$.

To find the fluent of this quantity, let $\frac{z^{-n-1}\dot{z}}{\overline{z^{-n}\pm 1}\rceil^2} = \dot{x}$, and $z^{m-n+1} = y$; then, since the fluent of $y\dot{x} = yx - f.\, x\dot{y}$, the fluent $= \frac{1}{n\times : z^{-n}\pm 1} \times z^{m-n+1} - f.\, \frac{m-n+1}{n} \times \frac{z^{m-n}\dot{z}}{z^{-n}\pm 1} = \frac{z^{m-n+1}}{n\times : z^{-n}\pm 1} - \frac{m-n+1}{n} \times f.\, \frac{z^m\dot{z}}{1\pm z^n} = \frac{z^{m+1}}{n\times \overline{1\pm z^n}} - \frac{m-n+1}{n} \times p.$

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## SECTION VI.

(335.) *On the Fluxions of Quantities which have a variable Index; and on the Fluents of Quantities which involve Quantities and their Logarithms, Arcs and their Sines, &c.*

### EXAMPLES.

(336.) Ex. 1. To find the fluxion of $x^y$.

Assume $x^y = v$; then $y \times$ hyp. log. $x =$ hyp. log. $v$; $\therefore \dot{y} \times$ hyp. log. $x + \frac{y\dot{x}}{x} = \frac{\dot{v}}{v}$; $\therefore \dot{v} = v\dot{y} \times$ hyp. log. $x + \frac{vy\dot{x}}{x} = x^y\dot{y}$ $\times$ hyp. log. $x + yx^{y-1}\dot{x}$.

COR. If $x$ be constant, $\dot{x} = 0$; and the fluxion $= x^y y \times$ hyp. log. $x$.

(337.) Ex. 2. The fluxion of $\overline{a^2+z^2}|^z = \overline{a^2+z^2}|^z \times \dot{z} \times$ hyp. log. $\overline{a^2+z^2} + z \times \overline{a^2+z^2}|^{z-1} \times 2z\dot{z}$. This appears by substituting for $x$ and its fluxion, in the last Case, the quantity $a^2+z^2$ and its fluxion.

(338.) Ex. 3. To find the fluxion of ${x^y}^z$.

Let $x^y = v$; then ${x^y}^z = v^z$; assume $v^z = w$; $\therefore$ by Art. 336. $\dot{w} = v^z\dot{z} \times$ hyp. log. $v + zv^{z-1}\dot{v}$; but $v = x^y$; $\therefore$ substitute for $v$ and its fluxion, as found in Art. 336., and we have $\dot{w} = {x^y}^z \dot{z} \times$ hyp. log. $x^y + z{x^y}^{z-1} \times : x^y\dot{y} \times$ hyp. log. $x + yx^{y-1}\dot{x} = {x^y}^z \dot{z} \times$ hyp. log. $x^y + z{x^y}^{z-1} x^y\dot{y} \times$ hyp. log. $x + yz{x^y}^{z-1} x^{y-1}\dot{x}$.

The quantity ${x^y}^z$ means the $z$ power of $x^y$. If it denoted the $y^z$ power of $x$, we must have assumed the index $y^z = v$.

In the same manner the fluxion may be found for any number of quantities.

### (339.) FLUENT 116.

To find the fluent of $zy\dot{y}$, where $z$ is a circular arc of radius 1, and sine $y$.

The fluent $= \frac{zy^2}{2} - f. \frac{y^2}{2} \times \dot{z} = \frac{zy^2}{2} - f. \frac{y^2}{2} \times \frac{\dot{y}}{\sqrt{1-y^2}} = \frac{zy^2}{2} - \frac{1}{4} \times$ a circular arc of radius 1, and sine $y + \frac{y \times \sqrt{1-y^2}}{4}$ (Fluent 15.)

### (340.) FLUENT 117.

To find the fluent of $\frac{2y\dot{y}}{\sqrt{1-y^2}}$, where $z$ is the arc of a circle, and $y$ the sine.

Let $\frac{y\dot{y}}{\sqrt{1-y^2}} = \dot{v}$; then the given fluxion $= z\dot{v}$, and the fluent $= zv - f. v\dot{z} = -z \times \sqrt{1-y^2} + f. \sqrt{1-y^2} \times \dot{z}$. Now

$\dot z = \frac{\dot y}{\sqrt{1-y^2}}$ to radius 1; ∴ $\sqrt{1-y^2} \times \dot z = \dot y$; and the whole fluent $= -z \times \sqrt{1-y^2} + y$.

### (341.) FLUENT 118.

To find the fluent of $\frac{z\dot t}{\overline{1+t^2}|^{\frac{3}{2}}}$, where $z$ is a circular arc of radius 1, and tangent $t$.

Here $\dot z = \frac{\dot t}{1+t^2}$; hence, by the Rule, p. 220, $f.\frac{z\dot t}{\overline{1+t^2}|^{\frac{3}{2}}}$ $= \frac{zt}{\sqrt{1+t^2}}$ (Fluent 40.) $- \frac{t}{\sqrt{1+t^2}} \times \frac{\dot t}{1+t^2} = \frac{zt}{\sqrt{1+t^2}} -$ $f.\frac{t\dot t}{\overline{1+t^2}|^{\frac{3}{2}}} = \frac{zt}{\sqrt{1+t^2}} + \frac{1}{\sqrt{1+t^2}}$.

(342.) The fluent of $\frac{z\dot x}{\overline{1-x^2}|^{\frac{3}{2}}}$, where $z$ is a circular arc of radius 1, and sine $x$, by a similar process, $= \frac{zx}{\sqrt{1-x^2}}$ $- f.\frac{x\dot x}{1-x^2} = \frac{zx}{\sqrt{1-x^2}}$ + hyp. log. $\sqrt{1-x^2}$.

### (343.) FLUENT 119.

To find the fluent of $\frac{z\dot x}{\overline{2x-x^2}|^{\frac{3}{2}}}$, where $z$ = a circular arc of radius 1, and versed sine $x$.

Here $\dot z = \frac{\dot x}{\sqrt{2x-x^2}}$. Let $\frac{\dot x}{\overline{2x-x^2}|^{\frac{3}{2}}} = \dot y$; then $z\dot y =$ $\frac{z\dot x}{\overline{2x-x^2}|^{\frac{3}{2}}}$; and $f.\, z\dot y = zy - f.\, y\dot z = \frac{z \times \overline{x-1}}{\sqrt{2x-x^2}}$ (Fluent 71.) $- f.\frac{x-1}{\sqrt{2x-x^2}} \times \frac{\dot x}{\sqrt{2x-x^2}} = \frac{z \times \overline{x-1}}{\sqrt{2x-x^2}} - f.\frac{x\dot x - \dot x}{2x-x^2} = \frac{z \times \overline{x-1}}{\sqrt{2x-x^2}}$ $- f. -1 \times \frac{\dot x - x\dot x}{2x-x^2} = \frac{z \times \overline{x-1}}{\sqrt{2x-x^2}}$ + hyp. log. $\sqrt{2x-x^2}$.

### (344.) FLUENT 120.

To find the fluent of $z y^n \dot{y}$, where $z$ is the arc of a circle, and $y$ the sine.

This fluent $= \frac{zy^{n+1}}{n+1} - f.\,\frac{y^{n+1}}{n+1} \times \dot{z}$; but $\dot{z} = \frac{r\dot{y}.}{\sqrt{r^2-y^2}}$ $= \dot{y} \times \overline{1-\frac{y^2}{r^2}}\Big|^{-\frac{1}{2}} =$ (by the binomial theorem) $\dot{y} \times : 1 + \frac{y^2}{2r^2} + \frac{y^4}{8r^4} +$ &c.; $\therefore f.\,\frac{y^{n+1}\dot{z}}{n+1} = \frac{1}{n+1} \times : y^{n+1}\dot{y} + \frac{y^{n+3}\dot{y}}{2r^2} + \frac{y^{n+5}\dot{y}}{8r^4} +$ &c. And the whole fluent $= \frac{zy^{n+1}}{n+1} - \frac{1}{n+1} \times : \frac{y^{n+2}}{n+2} + \frac{y^{n+4}}{\overline{n+4}\,.\,2r^2} + \frac{y^{n+6}}{\overline{n+6}\,.\,8r^4} +$ &c.

### (345.) FLUENT 121.

To find the fluent of $z^n \dot{y}$, where $z =$ the arc of a circle, and $y$ the corresponding sine.

Let $r =$ the radius, and $x$ the cosine; then, by Art. 44, $r\dot{y} = x\dot{z}$, and $y\dot{z} = -r\dot{x}$.

Assume $v = z^n$; then, since the fluent of $v\dot{y} = vy - f.\,y\dot{v}$, we have the fluent of $z^n\dot{y} = z^n y - f.\,y \times nz^{n-1}\dot{z}$; but $y\dot{z} = -r\dot{x}$; $\therefore$ the second part $= +f.\,nrz^{n-1}\dot{x}$; and the fluent of this part $= nz^{n-1}rx - f.\,n\,.\,\overline{n-1}\,.\,rxz^{n-2}\dot{z}$; and since $x\dot{z} = r\dot{y}$, the fluent of the last term $= -n\,.\,\overline{n-1}\,.\,z^{n-2}r^2y +$ &c. Hence the required fluent $= z^n y + nz^{n-1}rx - n\,.\,\overline{n-1}\,.\,z^{n-2}r^2y -$ &c.

(346.) The same fluent may be found in the following manner:

Assume $y \times : az^n + bz^{n-1} + cz^{n-2} +$ &c. $= f.\,z^n\dot{y}$;

$$\left.\begin{array}{l} \text{then } \dot{y} \times : az^n + bz^{n-1} + cz^{n-2} + \text{\&c.} \\ \quad + nayz^{n-1}\dot{z} + \overline{n-1}\,.\,byz^{n-2}\dot{z} + \text{\&c.} \end{array}\right\} = z^n\dot{y}.$$

Multiply $\dot{y}$ into each term of the first series, transpose $z^n\dot{y}$, and equate the corresponding coefficients; then $a\dot{y} - \dot{y} = 0$, $b\dot{y} + nay\dot{z} = 0$, $c\dot{y} + \overline{n-1}\,.\,by\dot{z} = 0$, &c; hence $a = 1$, $b\dot{y} =$

$-nay\dot z = -ny\dot z = nr\dot x$; $\therefore by = nrx$, and $b = \frac{nrx}{y}$, $c\dot y = -\overline{n-1}\,.\,by\dot z = -n\,.\,\overline{n-1} \times \frac{rx}{y} \times y\dot z = -n\,.\,\overline{n-1}\,.\,r^2\dot y$; $\therefore c = -n\,.\,\overline{n-1}\,.\,r^2$, &c.; and the fluent $= y \times : z^n + nz^{n-1}\frac{rx}{y} - n\,.\,\overline{n-1}\,z^{n-2}r^2 -$ &c. $= z^n y + nz^{n-1}rx - n\,.\,\overline{n-1}\,z^{n-2}r^2 y -$ &c. as before.

### (347.) FLUENT 122.

To find the fluent of $z^n\dot w$, where $w$ is the versed sine, corresponding to the arc $z$.

If $x$ be the cosine, $z^n\dot w = -z^n\dot x$. To find the fluent of $-z^n\dot x$, assume $x \times : az^n + bz^{n-1} + cz^{n-2} +$ &c. = the fluent; take the fluxion, and proceed as before; the required fluent will be $= -xz^n + nrz^{n-1}y + n\,.\,\overline{n-1}\,.\,r^2z^{n-2}x -$ &c.

### (348.) FLUENT 123.

To find the fluent of $zt^n\dot t$, where $z$ is a circular arc of radius $r$, and tangent $t$.

The fluent, by Sect. II., $= \frac{z\,t^{n+1}}{n+1} - f.\,\frac{t^{n+1}}{n+1} \times \dot z$. Now $\dot z = \frac{r^2\dot t}{r^2+t^2}$ (Art. 44.); therefore the second term of the fluent $= -f.\,\frac{r^2}{n+1} \times \frac{t^{n+1}\dot t}{r^2+t^2}$.

This admits of two cases:

1. Suppose $n$ to be an odd number; divide the numerator of this fraction by the denominator in an inverted order, as in Fluents 6 and 8, the result is $-\frac{r^2}{n+1} \times : t^{n-1}\dot t - r^2t^{n-3}\dot t + r^4t^{n-5}\dot t -$ &c. ± the remainder $\frac{r^{n+1}\dot t}{r^2+t^2}$; the number of terms in the quotient is $\frac{n+1}{2}$, and the remainder is positive or negative

according as $\frac{n+1}{2}$ is even or odd. Hence the whole fluent = $\frac{z\,t^{n+1}}{n+1} - \frac{r^2}{n+1} \times : \frac{t^n}{n} - \frac{r^2 t^{n-2}}{n-2} + \&c. \pm r^{n-1} \times$ a circular arc of radius $r$, and tangent $t$.

2. Let $n$ be an even number; the number of terms in the quotient will be $\frac{n}{2}$, and the remainder $\pm f.\, \frac{r^n t\dot{t}}{r^2+t^2}$, the sign of the remainder being + or −, according as $\frac{n}{2}$ is even or odd. The rest of the fluent is the same as before, and the $f.\, \frac{r^n t\dot{t}}{r^2+t^2}$ $= r^n \times$ hyp. log. $\sqrt{r^2+t^2}$.

### (349.) FLUENT 124.

To find the fluent of $vx\dot{x}$, where $v =$ hyp. log. $x + \sqrt{x^2+a^2}$.

In this case $\dot{v} = \frac{\dot{x}}{\sqrt{x^2+a^2}}$; and the fluent of $vx\dot{x}$, by Sect. II. Art. 211, $= \frac{vx^2}{2} - f.\, \frac{x^2}{2} \times \dot{v}$, or $- f.\, \frac{\frac{1}{2}\, x^2\dot{x}}{\sqrt{x^2+a^2}}$; hence, by Fluent 14, the whole fluent of $vx\dot{x} = \frac{vx^2}{2} - \frac{1}{4} \times x \times \sqrt{x^2+a^2} + \frac{1}{4}\, a^2 \times$ hyp. log. $x + \sqrt{x^2+a^2}$.

### (350.) FLUENT 125.

To find the fluent of $z^n\dot{x}$, where $z =$ hyp. log. $x$.

This $= z^n x - f.\, x \times nz^{n-1}\dot{z}$; or since $\dot{z} = \frac{\dot{x}}{x}$, it $= z^n x - f.\, nz^{n-1}\dot{x} = z^n x - nz^{n-1}x + n\,.\,\overline{n-1}\,.\,z^{n-2}x - \&c.$, where the law of continuation is manifest.

### (351.) FLUENT 126.

To find the fluent of $\frac{z\dot{x}}{\overline{2x+x^2}|^{\frac{3}{2}}}$, where $z$ is the hyp. log. of $1 + x + \sqrt{2x+x^2}$.

Here $\dot{z} = \frac{\dot{x}}{\sqrt{2x+x^2}}$; let $\dot{y} = \frac{\dot{x}}{\overline{2x+x^2}|^{\frac{3}{2}}}$; then $y = -\frac{x+1}{\sqrt{2x+x^2}}$;

$\therefore f.\, z\dot{y} = zy - f.\, y\dot{z} = -\frac{zx+z}{\sqrt{2x+x^2}} - f.\, -\frac{x+1}{\sqrt{2x+x^2}} \times \frac{\dot{x}}{\sqrt{2x+x^2}} = -\frac{zx+z}{\sqrt{2x+x^2}} + f.\, \frac{\dot{x}+x\dot{x}}{2x+x^2} = -z \times \frac{x+1}{\sqrt{2x+x^2}} +$ hyp. log. $\sqrt{2x+x^2}$.

## (352.) FLUENT 127.

To find the fluent of $\frac{z\dot{x}}{\overline{1+x^2}|^{\frac{3}{2}}}$, where $z =$ hyp. log. $x+\sqrt{1+x^2}$.

Here $\dot{z} = \frac{\dot{x}}{\sqrt{1+x^2}}$; also if $\frac{\dot{x}}{\overline{1+x^2}|^{\frac{3}{2}}} = \dot{y}$, then $y = \frac{x}{\sqrt{1+x^2}}$ (Fluent 40.); $\therefore$ the fluent $= \frac{zx}{\sqrt{1+x^2}} - f.\, \frac{x}{\sqrt{1+x^2}} \times \frac{\dot{x}}{\sqrt{1+x^2}} = \frac{zx}{\sqrt{1+x^2}} - f.\, \frac{x\dot{x}}{1+x^2} = \frac{zx}{\sqrt{1+x^2}} -$ hyp. log. $\sqrt{1+x^2}$.

## (353.) FLUENT 128.

To find the fluent of $\frac{z\dot{x}}{\overline{1-x^2}|^{\frac{3}{2}}}$, where $z =$ hyp. log. $\frac{1+x}{1-x}$.

Here $\dot{z} = \frac{2\dot{x}}{1-x^2}$; and if $\dot{y} = \frac{\dot{x}}{\overline{1-x^2}|^{\frac{3}{2}}}$, $y = \frac{x}{\sqrt{1-x^2}}$ (Fluent 40.); $\therefore f.\, z\dot{y} = \frac{zx}{\sqrt{1-x^2}} - f.\, \frac{2x\dot{x}}{\overline{1-x^2}|^{\frac{3}{2}}} = \frac{zx}{\sqrt{1-x^2}} - \frac{2}{\sqrt{1-x^2}} = \frac{zx-2}{\sqrt{1-x^2}}$.

## (354.) FLUENT 129.

To find the fluent of $\frac{\dot{x}}{1-x} \times$ hyp. log. $x$.

By Sect. II. fluent $= \text{hyp. log.} \frac{1}{1-x} \times \text{hyp. log.}\ x - f. \frac{\dot{x}}{x} \times \text{hyp. log.} \frac{1}{1-x}$. Now since the fluxion of the hyp. log. of $\frac{1}{1-x} = \frac{\dot{x}}{1-x} = \dot{x} + x\dot{x} + x^2\dot{x} + \text{\&c.}$; the hyp. log. $\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \text{\&c.}$; $\therefore f. \frac{\dot{x}}{x} \times \text{hyp. log.} \frac{1}{1-x} = f. \dot{x} + \frac{x\dot{x}}{2} + \frac{x^2\dot{x}}{3} + \text{\&c.}$; hence the whole fluent required $= \text{hyp. log.} \frac{1}{1-x} \times \text{hyp. log.}\ x - x - \frac{x^2}{4} - \frac{x^3}{9} - \text{\&c.}$

### (355.) FLUENT 130.

To find the fluent of $\frac{\dot{x}}{\sqrt{\text{hyp. log.} \frac{x}{a}}}$.

Let hyp. log. of $\frac{x}{a} = v$; then if $e$ be the number, whose hyp. log. is unity, $\frac{x}{a} = e^v$.

Also $x = ae^v$; and $\frac{\dot{x}}{x} = \dot{v}$; hence $\frac{\dot{x}}{\sqrt{\text{hyp. log.} \frac{x}{a}}} = \frac{ae^v\dot{v}}{\sqrt{v}}$.

Now by COR. to Art. 144, $e^v = 1 + v + \frac{v^2}{2} + \frac{v^3}{2 . 3} + \text{\&c.}$; $\therefore \frac{ae^v\dot{v}}{v^{\frac{1}{2}}} = av^{-\frac{1}{2}}\dot{v} + av^{\frac{1}{2}}\dot{v} + \frac{av^{\frac{3}{2}}\dot{v}}{2} + \frac{av^{\frac{5}{2}}\dot{v}}{2 . 3} + \text{\&c.}$; $\therefore$ the fluent required $= 2av^{\frac{1}{2}} + \frac{2av^{\frac{3}{2}}}{3} + \frac{2av^{\frac{5}{2}}}{5 . 2} + \text{\&c.}$

### (356.) FLUENT 131.

To find the fluent of $vx^n\dot{x}$, where $v = \text{hyp. log.} \frac{1}{x}$.

Here $\dot{v} = -\frac{\dot{x}}{x}$. Also if $x^n\dot{x} = \dot{w}$, since the $f.\, v\dot{w} = vw - f.\, w\dot{v}$, we have the $f.\, vx^n\dot{x} = \frac{vx^{n+1}}{n+1} - f.\, \frac{x^{n+1}}{n+1} \times \dot{v}$. Now $\dot{v} = -\frac{\dot{x}}{x}$; ∴ the last term $= +f.\, \frac{x^{n+1} \times \dot{x}}{n+1\,.\,x} = f.\, \frac{x^n\dot{x}}{n+1} = \frac{x^{n+1}}{\overline{n+1}|^2}$; hence the whole $= \frac{vx^{n+1}}{n+1} + \frac{x^{n+1}}{\overline{n+1}|^2}$.

### (357.) FLUENT 132.

To find the fluent of $vx^n\dot{x}$, where $v =$ hyp. log. $\overline{1-x}$.

This, by the method of Sect. II., $= \frac{vx^{n+1}}{n+1} - f.\, \frac{x^{n+1}\dot{v}}{n+1}$. Now $\dot{v} = \frac{-\dot{x}}{1-x}$; ∴ the second term $= +\frac{1}{n+1} \times f.\, \frac{x^{n+1}\dot{x}}{1-x}$; or, by actual division, $= \frac{1}{n+1} \times : -x^n\dot{x} - x^{n-1}\dot{x}$ — &c. to $n+1$ terms, and the remainder is $+\frac{\dot{x}}{1-x}$; ∴ the whole fluent $= \frac{vx^{n+1}}{n+1} - \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1} + \frac{x^n}{n}$ + &c. to $n+1$ terms, + hyp. log. $\overline{1-x}$.

### (358.) FLUENT 133.

By the same process the fluent of $vx^n\dot{x}$, where $v =$ hyp. log. $\frac{1}{1-x} = \frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1} + \frac{x^n}{n}$ + &c. to $n+1$ terms, − hyp. log. $\frac{1}{1-x}$.

### (359.) FLUENT 134.

And if $v = \frac{1}{2}$ hyp. log. $\frac{1+x}{1-x}$, the fluent of $vx^n\dot{x} = \frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times : \frac{x^n}{n} + \frac{x^{n-2}}{n-2} + \frac{x^{n-4}}{n-4}$ + &c. + $\frac{1}{2}$ hyp. log. $\overline{1-x^2}$, if $n$ be an even number; or $-\frac{1}{2}$ hyp. log. $\frac{1+x}{1-x}$, if $n$ be odd.

If $n$ be an even number, $n+1$ is odd, and the number of terms in the quotient is $\frac{n}{2}$ (Fluent 8.) If $n$ be odd, $n+1$ is even, and the number of terms is $\frac{n+1}{2}$ (Fluent 6).

### (360.) FLUENT 135.

To find the fluent of $z^m x^{n-1}\dot{x}$, where $z =$ hyp. log. $x$.

Assume the fluent $= x^n \times : az^m + bz^{m-1} + cz^{m-2} + \&c.$; then

$$\left.\begin{array}{l} nx^{n-1}\dot{x} \times : az^m + bz^{m-1} + cz^{m-2} + \&c. \\ + max^n z^{m-1}\dot{z} + \overline{m-1} \,.\, bx^n z^{m-2}\dot{z} + \&c. \end{array}\right\} = z^m x^{n-1}\dot{x}.$$

Transpose $z^m x^{n-1}\dot{x}$, and equate the corresponding terms; then $na\dot{x} - \dot{x} = 0$; $nb\dot{x} + max\dot{z} = 0$; $nc\dot{x} + \overline{m-1} \,.\, bx\dot{z} = 0$, &c.; $\therefore\ a = \frac{1}{n}$; $b = -\frac{m}{n^2}$; for $\dot{z} = \frac{\dot{x}}{x}$; $c = \frac{m \,.\, \overline{m-1}}{n^3}$, &c. Hence the fluent $= x^n \times : \frac{z^m}{n} - \frac{mz^{m-1}}{n^2} + \frac{m \,.\, m-1 \,.\, z^{m-2}}{n^3} - \&c.$, and the law of continuation is evident.

### (361.) FLUENT 136.

By the same process the fluent of $z^m x^{n-1}\dot{x}$, where $z =$ hyp. log. $-x$, may be found; the result is the same.

### (362.) FLUENT 137.

If in Fluent 135. $m=2$, and $n=1$, the fluxion becomes $z^2\dot{x}$, where $z =$ hyp. log. $x$. This is found immediately by Sect. II. $= z^2x - f. \, 2xz\dot{z} = z^2x - f. \, 2z\dot{x}$, since $\dot{z} = \frac{\dot{x}}{x} = z^2x - 2zx + f. \, 2x\dot{z} = z^2x - 2zx + 2x$.

### (363.) FLUENT 138.

To find the fluent of $Q^x x^n \dot{x}$.

Assume the fluent $= Q^x \times \overline{ax^n + bx^{n-1} + cx^{n-2} + \&c.}$, where $a$, $b$, $c$, &c. are constant; then, if $m =$ hyp. log. $Q$, we have (by Art. 336.) $$\left.\begin{array}{r} mQ^x\dot{x}\times : ax^n+bx^{n-1}+cx^{n-2}+\&c. \\ Q^x\dot{x}\times : nax^{n-1}+\overline{n-1}.b.x^{n-2}+\&c. \end{array}\right\} = Q^x x^n\dot{x};$$

$\therefore$ if both sides of the equation be divided by $Q^x\dot{x}$, and $x^n$ be transposed, $$\left.\begin{array}{r} max^n+mbx^{n-1}+mcx^{n-2}+\&c. \\ -x^n+nax^{n-1}+\overline{n-1}.bx^{n-2}+\&c. \end{array}\right\} = 0.$$

Equate the coefficients of the corresponding terms; then $ma-1=0$, $mb+na=0$, $mc+\overline{n-1}.b=0$; hence $a=\frac{1}{m}$, $b=-\frac{na}{m}=-\frac{n}{m^2}$, $c=-\frac{\overline{n-1}.b}{m}=\frac{n.\overline{n-1}}{m^3}$, &c.; $\therefore$ the fluent $= Q^x\times : \frac{x^n}{m}-\frac{nx^{n-1}}{m^2}+\frac{n.\overline{n-1}.x^{n-2}}{m^3}-$ &c., and the law of continuation is evident.

The same result would be obtained by assuming $Q^x\dot{x}=\dot{y}$ and $x^n=z$, and finding the fluent by the Rule in Sect. II.

## CHAP. XXIII.

### ON FLUXIONAL EQUATIONS.

(364.) THE object of this Chapter is, to shew the method of deducing from a fluxional equation the equation of the fluents, which may be called the primitive equation.

The fluxional equation, which expresses the value of $\frac{\dot{y}}{\dot{x}}$, deduced from a primitive, involving $y$ and $x$, is an equation of the first order; from this may be derived another, involving $\frac{\ddot{y}}{\dot{x}^2}$, this is of the second order; and a similar mode of definition may be used in the successive fluxions.

(365.) An equation is said to be homogeneous, when the sum of the exponents of the variable quantities $x$ and $y$ is the same in each term; as in the equation,

$$ax\dot{x} + by\dot{x} + dx\dot{y} + cy\dot{y} = 0,$$

$$\text{or } \overline{ax + by} \times \dot{x} + \overline{dx + cy} \times \dot{y} = 0.$$

(366.) The simplest case of fluxional equations is that in which the equation contains only one of the variable quantities with its fluxion in each term; here the fluent of each term must be taken separately.

### EXAMPLES.

Ex. 1. Let $ax^2\dot{x} = y^3\dot{y}$; then $\frac{ax^3}{3} = \frac{y^4}{4}$.

Ex. 2. Let $x^n y^m \dot{x} = a\dot{y}$, or $x^n\dot{x} = ay^{-m}\dot{y}$; then $\frac{x^{n+1}}{n+1} = \frac{ay^{1-m}}{1-m}$.

Ex. 3. $\dot{x} = \frac{2b^2c\dot{y}}{\sqrt{ay^3}} + \frac{3y^2\dot{y}}{a+b} + \dot{y} \times \sqrt{by+cy}$; that is, $\dot{x} = \frac{2b^2c}{\sqrt{a}} \times y^{-\frac{3}{2}}\dot{y} + \frac{3y^2\dot{y}}{a+b} + y^{\frac{1}{2}}\dot{y} \times \sqrt{b+c}$; $\therefore\ x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \sqrt{b+c} \times \frac{2y^{\frac{3}{2}}}{3}$.

Ex. 4. Given $\frac{\dot{x}}{\sqrt{2ax-x^2}} = \frac{\dot{y}}{\sqrt{b^2-y^2}}$; to find the nature of the curve, $x$ and $y$ being supposed to vanish together.

Here $\frac{1}{a} \times \frac{a\dot{x}}{\sqrt{2ax-x^2}} = \frac{1}{b} \times \frac{b\dot{y}}{\sqrt{b^2-y^2}}$; $\therefore$ by taking the fluents, $\frac{1}{a} \times$ a circular arc of radius $a$, and versed sine $x = \frac{1}{b} \times$ an arc of radius $b$, and sine $y$; $\therefore$ an arc, whose radius is 1, and versed sine $\frac{x}{a}$, = an arc whose radius is 1, and sine $\frac{y}{b}$. But since these radii are equal, and their arcs equal, the sines must be equal. Now the sine of an arc, whose radius is 1, and versed sine $\frac{x}{a}$, $= \frac{\sqrt{2ax-x^2}}{a}$; $\therefore \frac{\sqrt{2ax-x^2}}{a} = \frac{y}{b}$; and $y = \frac{b}{a} \times \sqrt{2ax-x^2}$, an equation to the ellipse.

(367.) The preceding rules will sometimes apply when two or more variable quantities are involved in the equation.

Ex. 5. If $\dot{v} = y^2z^3\dot{x} + 2xz^3y\dot{y} + 3xy^2z^2\dot{z}$, then $v = xy^2z^3$.

Ex. 6. If $\dot{v} = \frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$, then, by reducing the two terms on the right to a common denominator, $\dot{v} = \frac{y\dot{x} - x\dot{y}}{y^2}$, and $v = \frac{x}{y}$.

(368.) In some cases the quantities and their fluxions are so involved, that these methods cannot be applied. The equations of this class are not reducible to any general form; but the variable quantities may frequently be separated by some of the following rules.

(369.) Rule 1. Substitute for the sum of the two variable quantities, for the sum of their squares, for their products, &c.

Ex. 7. Let $a^2\dot{x}=\overline{x+y}|^2\times\dot{y}$.

Assume $x+y=z$; then $\dot{x}+\dot{y}=\dot{z}$, and $\dot{x}=\dot{z}-\dot{y}$; $\therefore$ by substitution, $a^2\dot{z}-a^2\dot{y}=z^2\dot{y}$, and $\dot{y}=\frac{a^2\dot{z}}{a^2+z^2}$, whose fluent is known by Art. 44.

Ex. 8. Let $\overline{x\dot{y}+y\dot{x}}\times\sqrt{a^4-x^2y^2}=\frac{x\dot{x}+y\dot{y}}{\overline{x^2+y^2}|^{\frac{3}{4}}}$.

Assume $xy=z$, and $x^2+y^2=v$; then $x\dot{y}+y\dot{x}=\dot{z}$, and $x\dot{x}+y\dot{y}=\frac{\dot{v}}{2}$; $\therefore\ \dot{z}\times\sqrt{a^4-z^2}=\frac{\dot{v}}{2v^{\frac{3}{4}}}$, the fluents of which are known.

(370.) Rule 2. Multiply or divide the given equation by some function of the unknown quantities, so as to bring it to a form which is known.

Ex. 9. Let $my\dot{x}+ny\dot{x}=0$.

Divide the terms by $xy$; then $\frac{m\dot{x}}{x}+\frac{n\dot{y}}{y}=0$; $\therefore\ m\times$ hyp. log. $x+n\times$ hyp. log. $y=$ some given quantity $=c$.

Ex. 10. Let $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}=\frac{ax^m\dot{x}}{y^n}$.

Multiply the equation by $nx^ny^n$; then $ny^nx^{n-1}\dot{x}+nx^ny^{n-1}\dot{y}$ $=nax^{m+n}\dot{x}$; and by taking the fluent, $nx^ny^n=\frac{nax^{m+n+1}}{m+n+1}$; and if no correction be wanted, $y^n=\frac{ax^{m+1}}{m+n+1}$.

Ex. 11. Let $\frac{p\dot{x}}{x} + \frac{r\dot{y}}{y} = \frac{x^m\dot{x}}{ay^r}$.

Multiply by $x^py^r$; then $px^{p-1}\dot{x} \times y^r + x^p \times ry^{r-1}\dot{y} = \frac{x^{m+p}\dot{x}}{ay^0}$. The fluent of the first part is $x^py^r$; and since $y^0 = 1$, the fluent of the second part $= \frac{x^{m+p+1}}{\overline{m+p+1} \times a}$; therefore, if there be no correction, $y^r = \frac{x^{m+1}}{\overline{m+p+1} \times a}$.

Ex. 12. Let $\frac{a\dot{x}}{x} + \frac{b\dot{y}}{y} - \frac{c\dot{z}}{z} = 0$.

Multiply by $\frac{x^ay^b}{z^c}$; then the resulting fluxion is $\frac{ax^{a-1}\dot{x}y^b}{z^c}$ $+ \frac{by^{b-1}\dot{y}x^a}{z^c} - \frac{cx^ay^b\dot{z}}{z^{c+1}} = 0$; and the fluent is $\frac{x^ay^b}{z^c} = A$.

The same method may be applied to fluxional equations of higher orders.

Ex. 13. Let $\ddot{x} - x\dot{z}^2 = b\dot{z}^2$.

Here $\ddot{z}$ is not found; therefore in deducing the fluxional equation, $\dot{z}$ must have been supposed constant. Multiply by $\dot{x}$; then $\dot{x}\ddot{x} - x\dot{x}\dot{z}^2 = b\dot{x}\dot{z}^2$; and as $\dot{z}$ is constant, the fluent is $\frac{\dot{x}^2}{2} - \frac{x^2\dot{z}^2}{2} = bx\dot{z}^2$; therefore $\dot{z} = \frac{\dot{x}}{\sqrt{2bx+x^2}}$, and $z =$ hyp. log. $\overline{b+x+\sqrt{2bx+x^2}}$.

(371.) RULE 3. It is sometimes useful to substitute for one of the unknown quantities the sum of the other, and a new variable quantity.

Ex. 14. Let $a\dot{z} = z\dot{x} - x\dot{x}$.

Assume $z = a+x+v$; then $\dot{z} = \dot{x}+\dot{v}$; ∴ by substituting for $z$ and $\dot{z}$, we have $a\dot{x}+a\dot{v} = a\dot{x}+x\dot{x}+v\dot{x}-x\dot{x}$, or $a\dot{v} = v\dot{x}$; $\therefore\ \dot{x} = \frac{a\dot{v}}{v}$, and $x = a \times$ hyp. log. $v$. Hence $z = a+v+a \times$ hyp. log. $v$. Substitute for $v$ its value, $z-a-x$, and $x = a \times$ hyp. log. $\overline{z-a-x}$.

(372.) **Rule 4.** Substitute for one fluxion or for one unknown quantity, the product of the other and some new variable quantity. This mode of substitution may be adopted in homogeneous equations.

Ex. 15. Let the equation be $\frac{\dot{y}}{y} - \frac{\dot{x}}{x} = \frac{x^3\dot{x}}{ay^2}$.

Assume $y=zx$; then $\dot{y}=z\dot{x}+x\dot{z}$. Hence, by substitution, $\frac{z\dot{x}+x\dot{z}}{zx} - \frac{\dot{x}}{x} = \frac{x^3\dot{x}}{az^2x^2}$, or $\frac{\dot{z}}{z} = \frac{x^3\dot{x}}{az^2x^2}$; ∴ $az\dot{z}=x\dot{x}$; and if no correction be wanted, $x^2=az^2=\frac{ay^2}{x^2}$, and $x^4=ay^2$.

Ex. 16. Let $x\dot{x}+ay\dot{x}+y\dot{y}=0$.

Assume $y=zx$; ∴ $\dot{y}=z\dot{x}+x\dot{z}$. Hence, by substitution, $\dot{x} + azx\dot{x} + z^2x\dot{x} + x^2z\dot{z}=0$; or $x\dot{x}\times\overline{1+az+z^2} = -x^2z\dot{z}$; ∴ $-\frac{\dot{x}}{x} = \frac{z\dot{z}}{1+az+z^2}$, whose fluent is known by Fluent 101.

Ex. 17. To find the fluent of $ax\dot{x}+by\dot{x}+dx\dot{y}+ey\dot{y}=0$.

Let $y=zx$; then $\dot{y}=z\dot{x}+x\dot{z}$; therefore, by substitution, $ax\dot{x} + bzx\dot{x} + dzx\dot{x} + dx^2\dot{z} + ez^2x\dot{x} + ex^2z\dot{z} = 0$. Assume $c=b+d$; then $x\dot{x}\times\overline{a+cz+ez^2} + \overline{d\dot{z}+ez\dot{z}}\times x^2=0$, or $\frac{\dot{x}}{x} = -\frac{d\dot{z}+ez\dot{z}}{a+cz+ez^2}$.

The fluent of the first part = hyp. log. of $x$; and that of the second is found by Fluents 100. and 101.

Ex. 18. Let $x\dot{y}-y\dot{x}=\dot{x}\times\sqrt{x^2+y^2}$.

Assume $y=zx$; then, by substituting for $y$ and $\dot{y}$, $zx\dot{x}+x^2\dot{z}-zx\dot{x}=\dot{x}\times\sqrt{x^2+z^2x^2}$, or $x^2\dot{z}=x\dot{x}\times\sqrt{1+z^2}$; therefore $\frac{\dot{z}}{\sqrt{1+z^2}}=\frac{\dot{x}}{x}$; and hyp. log. $\overline{z+\sqrt{1+z^2}}=$ hyp. log. $x$.

Ex. 19. Let $\frac{\dot{x}}{x} - \frac{\dot{y}}{y} = \frac{x^m\dot{x}}{ay^n}$.

Assume $y=zx$; then $\dot{y}=z\dot{x}+x\dot{z}$. Hence, by substitution, $\frac{\dot{x}}{x}-\frac{z\dot{x}+x\dot{z}}{zx}=\frac{x^m\dot{x}}{az^nx^m}$, or $-\frac{\dot{z}}{z}=\frac{x^m\dot{x}}{az^nx^m}$; that is, $-z^{n-1}\dot{z}$ $=\frac{x^{m-n}\dot{x}}{a}$; $\therefore$ Cor. $-\frac{z^n}{n}=\frac{x^{m-n+1}}{a\times\overline{m-n+1}}$; but $z=\frac{y}{x}$; $\therefore$ by substitution, Cor. $-\frac{y^n}{n}=\frac{x^{m+1}}{a\times\overline{m-n+1}}$.

(373.) **Rule 5.** If only one of the variable quantities be found in the equation, substitute for one fluxion the product of the other, and some new variable quantity.

Ex. 20. Let $a\dot{x}\dot{y}^3=y\times\overline{\dot{x}^2+\dot{y}^2}|^2$.

Let $\dot{x}=z\dot{y}$; then $az\dot{y}^4=y\times\overline{z^2\dot{y}^2+\dot{y}^2}|^2=y\dot{y}^4\times\overline{z^2+1^2}|^2$; $\therefore$ $y=\frac{az}{\overline{z^2+1}|^2}$; hence $\dot{y}=\frac{a\dot{z}-3az^2\dot{z}}{\overline{z^2+1}|^3}$; therefore $\dot{x}$ or $z\dot{y}=$ $\frac{az\dot{z}-3az^3\dot{z}}{\overline{z^2+1}|^3}$, whose fluent or $x$, by Fluent 55, $=\frac{3a}{2\times\overline{1+z^2}}$ $-\frac{a}{\overline{1+z^2}|^2}$.

Ex. 21. Let $-\frac{\ddot{y}}{\dot{x}^2+\dot{y}^2}=\frac{1}{y}$.

Here $\dot{x}$ is constant. Assume $\dot{y}=z\dot{x}$. Then $\ddot{y}=\dot{z}\dot{x}$; hence, by substitution, $-\frac{\dot{z}\dot{x}}{\dot{x}^2+z^2\dot{x}^2}=\frac{1}{f.\,z\dot{x}}$; $\therefore$ $-\dot{z}\dot{x}\times f.\,z\dot{x}$ $=\dot{x}^2+z^2\dot{x}^2$, or $-f.\,z\dot{x}=\frac{\overline{1+z^2}\times\dot{x}}{\dot{z}}$. Take the fluxion; then $-z\dot{x}=2z\dot{x}-\frac{\overline{1+z^2}\times\ddot{z}\dot{x}}{\dot{z}^2}$; $\therefore$ $\frac{\ddot{z}}{\dot{z}}=\frac{3z\dot{z}}{1+z^2}$; and hyp. log. $\dot{z}$ + Cor. $=\frac{3}{2}\times$ hyp. log. $\overline{1+z^2}$.

Since $\dot{z}$ or $\dot{x}$ must enter into every term, let $\dot{z}$ be corrected by taking it $=a\dot{x}$, a given quantity; then hyp. log. $\frac{\dot{z}}{a\dot{x}}=$ $\frac{3}{2}\times$ hyp. log. $\overline{1+z^2}$; $\therefore$ $\frac{\dot{z}}{a\dot{x}}=\overline{1+z^2}|^{\frac{3}{2}}$, and $a\dot{x}=\frac{\dot{z}}{\overline{1+z^2}|^{\frac{3}{2}}}$;

$\therefore z\dot{x}$ or $\dot{y} = \dfrac{z\dot{z}}{a \times \overline{1+z^2}|^{\frac{3}{2}}}$; hence (Fluent 40.) $x = \dfrac{z}{a \times \sqrt{1+z^2}}$ + Cor., and $y = -\dfrac{1}{a \times \sqrt{1+z^2}}$ + Cor.

Ex. 22. Let $y\dot{y}^3\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$.

Take $z\dot{y} = \dot{x}$; then $yz\dot{y}^4 = az^4\dot{y}^4 + 2az^2\dot{y}^4 + a\dot{y}^4$; $\therefore y = az^3 + 2az + \dfrac{a}{2}$, and $\dot{y} = 3az^2\dot{z} + 2a\dot{z} - \dfrac{a\dot{z}}{z^2}$. Now $\dot{x} = z\dot{y}$; $\therefore \dot{x} = 3az^3\dot{z} + 2az\dot{z} - \dfrac{a\dot{z}}{z}$, and $x = \dfrac{3az^4}{4} + az^2 - a \times$ hyp. log. $z$.

$x$ may be found in terms of $y$, by obtaining the value of $z$ from the equation $y = az^3 + 2az + \dfrac{a}{z}$, and substituting it in the equation $x = \dfrac{3az^4}{4} + az^2 - a \times$ hyp. log. $z$.

Ex. 23. Let $x = \dfrac{y\dot{x}}{\dot{y}} + \dfrac{ay^2\dot{x}^2}{\dot{y}^2} + \dfrac{by^3\dot{x}^3}{\dot{y}^3}$ + &c.

Assume $\dot{x} = z\dot{y}$; then, by substitution, $x = zy + az^2y^2 + bz^3y^3$ + &c.

In this equation, for $zy$ write $v$; then $x = v + av^2 + bv^3$ + &c.; $\therefore \dot{x}$ or $z\dot{y} = \dot{v} + 2av\dot{v} + 3bv^2\dot{v}$ + &c.; that is, $\dfrac{v\dot{y}}{y} = \dot{v} + 2av\dot{v} + 3bv^2\dot{v}$ + &c.; $\therefore$ hyp. log. $y$ = hyp. log. $v + 2av + \dfrac{3bv^2}{2}$ + &c.

(374.) RULE 6. Take the fluxion of the given equation, one of the fluxions being supposed constant.

Ex. 24. Let the equation be $y + \dfrac{a\dot{y} - x\dot{y}}{\dot{x}} = x - \dfrac{y\dot{x}}{\dot{y}}$.

Make $\dot{x}$ constant, and take the fluxion; then $\dot{y} + \dfrac{a\ddot{y} - x\ddot{y} - \dot{y}\dot{x}}{\dot{x}} = \dot{x} + \dfrac{\ddot{y}y\dot{x} - \dot{y}^2\dot{x}}{\dot{y}^2}$; $\therefore \ddot{y} \times \dfrac{\overline{a - x}}{\dot{x}} = \dfrac{\ddot{y}y\dot{x}}{\dot{y}^2}$, and

$\frac{\dot{y}^2}{y} = \frac{\dot{x}^2}{a-x}$; hence $y^{-\frac{1}{2}}\dot{y} = \overline{a-x}|^{-\frac{1}{2}} \times \dot{x}$; therefore $2y^{\frac{1}{2}} = -2 \times \overline{a-x}|^{\frac{1}{2}}$.

Ex. 25. The equation $\frac{a\dot{x}+y\dot{x}}{\dot{y}} = x + y - \frac{x\dot{y}}{\dot{x}}$, solved in the same manner, gives $2x^{\frac{1}{2}} = 2 \times \overline{a+y}|^{\frac{1}{2}}$.

Ex. 26. Let $y\ddot{y}\dot{x} - x\dot{y}^2 = a\dot{x}^2$.

Take $\dot{x}$ constant; then $\dot{y}^2\dot{x} + y\ddot{y}\dot{x} - \dot{x}\dot{y}^2 - 2x\dot{y}\ddot{y} = 0$, or $y\ddot{y}\dot{x} = 2x\dot{y}\ddot{y}$; $\therefore y\dot{x} = 2\dot{y}x$, or $\frac{\dot{x}}{x} = \frac{2\dot{y}}{y}$; hence hyp. log. $x = 2 \times$ hyp. log. $y$, and $ax = y^2$.

(375.) Rule 7. The equation may sometimes be brought to a better form by completing the square and extracting the root.

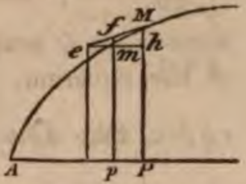
Ex. 27. Let $\dot{y}^2 = \dot{x}\dot{y} - x^2\dot{x}^2$.

Hence $\dot{y}^2 - \dot{x}\dot{y} + \frac{\dot{x}^2}{4} = \frac{\dot{x}^2}{4} - x^2\dot{x}^2$; $\therefore \dot{y} - \frac{\dot{x}}{2} = \pm\dot{x} \times \sqrt{\frac{1}{4} - x^2}$; $\therefore \dot{y} = \frac{\dot{x}}{2} \pm \dot{x} \times \sqrt{\frac{1}{4} - x^2}$, and $y = \frac{x}{2}$ ± the area *GPEA*, where $GP = x$, and $GE = \frac{1}{2}$. (Fig. p. 229.)

Ex. 28. Let the equation be $x^2\dot{x}^2 + xy\dot{x}\dot{y} = a^2\dot{y}^2$.

Complete the square; then $x^2\dot{x}^2 + x\dot{x}y\dot{y} + \frac{y^2\dot{y}^2}{4} = a^2\dot{y}^2 + \frac{y^2\dot{y}^2}{4}$; therefore, by extracting the square root, $x\dot{x} + \frac{y\dot{y}}{2} = \dot{y} \times \sqrt{a^2 + \frac{1}{4}y^2} = \frac{\dot{y}}{2} \times \sqrt{4a^2 + y^2} = \frac{2a^2\dot{y}}{\sqrt{4a^2+y^2}} + \frac{\frac{1}{2}y^2\dot{y}}{\sqrt{4a^2+y^2}}$; the fluents of which are known by Art. 43. and 214.

(376.) Prob. To transform a fluxional equation of the second order, containing two variable quantities $x$ and $y$, of which $x$ flows uniformly into another in which $y$ shall flow uniformly, or $\dot{y}$ be constant.



Let $MA$ be a curve, whose abscissa $AP$ flows with a uniform velocity; let $AP=x$, $Pp=\dot{x}$, $PM=y$, and $2mf=\ddot{y}$. Next, suppose $y$ to flow uniformly; let $Mh=\dot{y}$; then $he=\dot{x}$, and $2me=\ddot{x}$. Now by similar triangles, $Mhe$, $fme$, $eh\ (\dot{x}) : hM\ (\dot{y}) :: 2em\ (\ddot{x}) : 2mf\ (-\ddot{y})$; $\therefore \ddot{y} = -\dfrac{\dot{y}\ddot{x}}{\dot{x}}$, which, being substituted for $\ddot{y}$ in the original equation, will give another equation in which $\dot{y}$ is constant.

(377.) The same proposition may be thus demonstrated:

Let $y = a + bx + cx^2 + dx^3 +$ &c.; then $\dfrac{\dot{y}}{\dot{x}} = b + 2cx + 3dx^2 +$ &c.

Assume $\dot{x}$ constant, and take the fluxion; then $\dfrac{\ddot{y}}{\dot{x}} = 2c\dot{x} + 6dx\dot{x} +$ &c.

Next make $\dot{y}$ constant, and take the fluxion; then $-\dfrac{\dot{y}\ddot{x}}{\dot{x}^2} = 2cx + 6dx\dot{x} +$ &c.; $\therefore \dfrac{\ddot{y}}{\dot{x}}$, when $\dot{x}$ is constant, $= -\dfrac{\dot{y}\ddot{x}}{\dot{x}^2}$; when $\dot{y}$ is constant, or $\ddot{y}$, $= -\dfrac{\dot{y}\ddot{x}}{\dot{x}}$.

(378.) RULE 8. Any fluxional equation of the second order, containing only $x$ and $y$, in which $\dot{x}$ is constant, may be transformed into another in which $\dot{y}$ is constant, by substituting for $\ddot{y}$ the fraction $-\dfrac{\dot{y}\ddot{x}}{\dot{x}}$.

Ex. 29. Let $\dot{x}\dot{y} - x\ddot{y} - a\ddot{y} - \dfrac{x\dot{y}^2}{b} = 0$.

In this case, since $\ddot{x}$ is not found, $\dot{x}$ is constant. Substitute according to the Rule for $\ddot{y}$; then $\dot{x}\dot{y} + x \times \dfrac{\dot{y}\ddot{x}}{\dot{x}} + a \times \dfrac{\dot{y}\ddot{x}}{\dot{x}} - \dfrac{x\dot{y}^2}{b} = 0$, or $\dot{x}^2 + x\ddot{x} + a\ddot{x} - \dfrac{x\dot{x}\dot{y}}{b} = 0$; the fluent $= x\dot{x} + a\dot{x}$

$-\frac{x^2\dot{y}}{2b}$, which must be equal to some constant quantity. Now since $\dot{x}$ or $\dot{y}$ must be found in every term of the first fluxion of the equation, the constant quantity must be of the form $c\dot{y}$; $\therefore$ take $x\dot{x}+a\dot{x}-\frac{x^2\dot{y}}{2b}=c\dot{y}$; then $\dot{y}=\frac{2bx\dot{x}}{2bc+x^2}+\frac{2ab\dot{x}}{2bc+x^2}$,

and $y=b\times\text{hyp. log.}\ \overline{2bc+x^2}+\frac{a}{c}\times$ a circular arc, whose radius is $\sqrt{2bc}$, and tangent $x$.

(379.) This equation may also be solved in the following manner:

Multiply it by the fraction $\frac{\dot{x}}{\dot{y}^2}$; the result is $\frac{\dot{x}^2\dot{y}-x\dot{x}\ddot{y}}{\dot{y}^2}-\frac{a\dot{x}\ddot{y}}{\dot{y}^2}-\frac{x\dot{x}}{b}=0$; and the fluent $\frac{x\dot{x}}{\dot{y}}+\frac{a\dot{x}}{\dot{y}}-\frac{x^2}{2b}=$ some constant quantity $=c$; $\therefore x\dot{x}+a\dot{x}-\frac{x^2\dot{y}}{2b}=c\dot{y}$, the same as in the last case.

(380.) The most general method is that of an infinite series, which is to be used when other artifices fail:

RULE 9. For the quantity whose value is to be found, assume a series involving the powers of the other with unknown coefficients; substitute this series and its fluxions for their values in the given equation, and equate the coefficients of the corresponding terms.

(381.) Ex. 30. Let $\frac{m\dot{x}}{1+x}=\dot{y}$, or $m\dot{x}-\dot{y}-x\dot{y}=0$.

Assume the series, and follow the process adopted in Art. 144; $x=\frac{y}{m}+\frac{y^2}{2m^2}+$ &c.

Ex. 31. Let $y\dot{x}^2+m\dot{x}\dot{y}-m^2\ddot{y}=0$.

In this case $\dot{x}$ is constant. Assume $\dot{x}=1$, and let $y=ax+bx^2+cx^3+dx^4+$ &c.; then $\dot{y}=a+2bx+3cx^2+4dx^3+$ &c.,

and $\ddot{y} = 2b + 2 \,.\, 3 \,.\, cx + 3 \,.\, 4 \,.\, dx^2 +$ &c.; hence, by substitution in the original equation,

$$\left.\begin{array}{r} ax + bx^2 + \text{\&c.} \\ +ma + 2mbx + 3mcx^2 + \text{\&c.} \\ -2m^2b - 2 \,.\, 3m^2cx - 3 \,.\, 4 \,.\, m^2dx^2 - \text{\&c.} \end{array}\right\} = 0.$$

Hence $ma - 2m^2b = 0$, $a + 2mb - 2 \,.\, 3 \,.\, m^2c = 0$, $b + 3mc - 3 \,.\, 4 \,.\, m^2d = 0$, &c.; $\therefore b = \frac{ma}{2m^2} = \frac{a}{2m}$, $c = \frac{2a}{2 \,.\, 3 \,.\, m^2}$, &c.; $\therefore y = ax + \frac{ax^2}{2m} + \frac{2ax^3}{2 \,.\, 3m^2} +$ &c. $= a \times : x + \frac{x^2}{2m} + \frac{2x^3}{2 \,.\, 3m^2}$ + &c.

(382.) Ex. 32. Let $mx^2\dot{x} + y\dot{x} - n\dot{y} = 0$.

Assume $y = ax^3 + bx^4 + cx^5 +$ &c.; $\therefore \dot{y} = 3ax^2\dot{x} + 4bx^3\dot{x} + 5cx^4\dot{x} +$ &c. And by substitution in the given equation,

$$\left.\begin{array}{r} mx^2\dot{x} + ax^3\dot{x} + bx^4\dot{x} + \text{\&c.} \\ -3nax^2\dot{x} - 4nbx^3\dot{x} - 5ncx^4\dot{x} - \text{\&c.} \end{array}\right\} = 0.$$

Hence $m - 3na = 0$, $a - 4nb = 0$, $b - 5nc = 0$, &c.; $\therefore a = \frac{m}{3n}$, $b = \frac{a}{4n} = \frac{m}{3 \,.\, 4 \,.\, n^2}$, $c = \frac{b}{5n} = \frac{m}{3 \,.\, 4 \,.\, 5n^3}$, &c.; $\therefore y = \frac{mx^3}{3n} + \frac{mx^4}{3 \,.\, 4 \,.\, x^2} + \frac{mx^5}{3 \,.\, 4 \,.\, 5 \,.\, n^3} +$ &c.

If in this equation we had assumed $y = Ax + Bx^2 + Cx^3$ + &c., it would have appeared that $A = 0$, and $B = 0$; so that a regular series beginning with $x$ is not always required. The principal difficulty is to determine, what kind of series with respect to the indices ought to be taken, that no superfluous terms may be admitted. As the subject is curious rather than useful, the Reader is referred to *Simpson's Fluxions*, p. 296. Second Edition.

# CHAP. XXIV.

## SECT. I.

## PROBLEMS.

### PROB. 1.

Two weights are connected by a chord going over a single fixed pulley; to determine their ratio, so that one shall generate the greatest quantity of motion possible in the other in a given time.

Let $P$ be the greater weight, and $y$ the less; then the accelerating force $=\frac{P-y}{P+y}$; $\therefore$ the momentum of $y$ and $\propto \frac{\overline{P-y}}{P+y} \times y$, which is a maximum by the Problem; hence the fluxion of $\frac{Py-y^2}{P+y}=0$, or $\overline{P\dot{y}-2y\dot{y}} \times \overline{P+y}-\dot{y} \times \overline{Py-y^2}=0$, or $\overline{P-2y} \times \overline{P+y}=Py-y^2$; $\therefore y^2+2Py=P^2$; and by solving the quadratic, $y=P \times \overline{\sqrt{2}-1}$.

$$P : y :: 1 : \sqrt{2}-1.$$

### PROB. 2.

Materials are to be raised to a given height by means of a given weight hanging over a single fixed pulley? What weight must be raised each time, so that the greatest possible quantity may be raised in a given time?

Let $t$ represent the whole time, $n$ the time of one ascent, $S$ the given space; then $S=mn^2\times F$; $\therefore n=\sqrt{\frac{S}{mF}}$; but $\frac{t}{n}=$ the number of ascents; $\therefore$ this number $=t\times\sqrt{\frac{mF}{S}}$. Also, if $P$ be the given power, and $x$ the weight raised each time, $F=\frac{P-x}{P+x}$; $\therefore$ the number $=t\times\sqrt{\frac{m}{S}}\times\sqrt{\frac{P-x}{P+x}}$; and the whole weight raised $=t\times\sqrt{\frac{m}{S}}\times\sqrt{\frac{P-x}{P+x}}\times x\propto\sqrt{\frac{P-x}{P+x}}\times x$, which by the Problem is a maximum. Hence its square $\frac{Px^2-x^3}{P+x}$ is a maximum, and its fluxion $=0$; that is, $\overline{2Px\dot{x}-3x^2\dot{x}}\times\overline{P+x}-\dot{x}\times\overline{Px^2-x^3}=0$, from which equation $x=\frac{P\times\overline{\sqrt{5}-1}}{2}$.

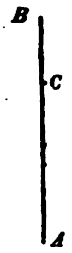
### Prob. 3.

A body falls from $A$ to $B$ by the force of gravity, and then rolls uniformly on a given horizontal line $BC$ with the velocity acquired; to find $AB$, so that the time of falling down $AB$ added to the time of describing $BC$ may be a minimum.

Let $AB=x$, $BC=b$, $m=16\frac{1}{12}$ feet. The time down $AB=\sqrt{\frac{x}{m}}$, and the velocity acquired $=\sqrt{4mx}$. Also, since the velocity in $BC$ is uniform, the time of describing $BC=\frac{BC}{v}=\frac{b}{\sqrt{4mx}}$; $\therefore$ by the Problem, $\sqrt{\frac{x}{m}}+\frac{b}{\sqrt{4mx}}$ is a minimum, or $x^{\frac{1}{2}}+\frac{bx^{-\frac{1}{2}}}{2}$ is a minimum; hence $\frac{\dot{x}}{2\sqrt{x}}-\frac{b\dot{x}}{4x^{\frac{3}{2}}}=0$, and $x=\frac{b}{2}$, or $AB=\frac{BC}{2}$.

### PROB. 4.

From what height above $C$ must a perfectly elastic ball be suffered to descend by gravity, so that it may impinge upon $A$, and return to the given point $C$, in the least time possible?



Let $B$ be the point required; take $BC=x$, $AC=a$; then the time through $BA=\sqrt{\frac{a+x}{m}}$; time through $BC=\sqrt{\frac{x}{m}}$; $\therefore$ the time down $CA$, or the time of rising through $AC$ after impact, $=\frac{\sqrt{a+x}-\sqrt{x}}{\sqrt{m}}$. Hence the whole time of falling through $BA$ and returning to $C=\frac{2\sqrt{a+x}-\sqrt{x}}{\sqrt{m}}$, a minimum. Hence $\frac{\dot{x}}{\sqrt{a+x}}-\frac{\dot{x}}{2\sqrt{x}}=0$, and $x$ is found $=\frac{a}{3}$.

### PROB. 5.

Given two perfectly elastic bodies $A$ and $B$; to find an intermediate elastic body $x$ of such magnitude, that the motion communicated from $A$ to $B$ through $x$ may be a maximum.

Let $a$ = the velocity communicated to $A$, $w$ the velocity communicated to $x$, and $b$ that communicated to $B$; then, by mechanics,

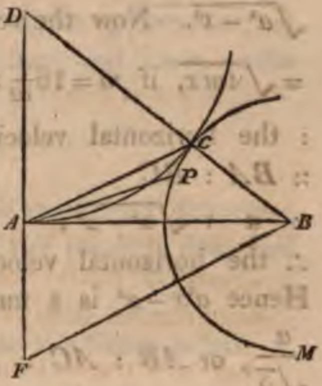
$$\begin{array}{l} A+x : 2A :: a : w \\ x+B : 2x :: w : b; \\ \hline \therefore \overline{A+x}\times\overline{x+B} : 4Ax :: a : b, \text{ or} \\ A+x+\frac{AB}{x}+B : 4A :: a : b. \end{array}$$

Now the two mean terms are constant; therefore the first varies inversely as the fourth; but the fourth is a maximum; therefore the first is a minimum. Hence, by taking the fluxion, $\dot{x}-\frac{AB\dot{x}}{x^2}=0$; $\therefore x^2=AB$, and $A : x :: x : B$, or $x$ is a mean proportional between $A$ and $B$.

Cor. If $A$ impinge upon $x$ at rest, $x$ upon $y$, $y$ upon $z$, &c. and $z$ upon $B$, the motion communicated to $B$ is a maximum, when the several bodies interposed are geometric means between the first and the last.

### Prob. 6.

Given two sides of a triangle; required the third, so that a body may fall down it by the force of gravity in the least time possible.



Let $AB$, $BC$ be the two given sides, of which the greater $AB$ is drawn parallel to the horizon; with $B$ as a center, and the less side $BC$ as a radius, describe the circular arc $CM$. From $A$ draw $AF$ perpendicular to $AB$, and make $AF=BC$; join $FB$. Make the angle $FBC$ equal to the angle $BFA$; produce $BC$ and $FA$ till they meet in $D$, and join $AC$; $CA$ is the line required.

For if $BC$ be produced to $D$, since $DF=DB$, and $AF=CB$; therefore $DA=DC$; and a circle described, with the center $D$ and radius $DA$, will touch $CM$ in $C$. Hence, by mechanics, the time down $CA$ is less than the time down any other line $PA$ drawn from $A$ to the circumference $CM$.

### Prob. 7.

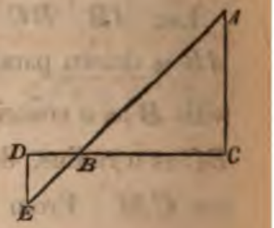
Given the base of an inclined plane; required its height, so that the time down the plane may be a minimum.

Let the base $= b$, the height $= x$; then the length $= \sqrt{a^2+x^2}$; but the time $\propto \frac{L}{\sqrt{H}} \propto \frac{\sqrt{a^2+x^2}}{\sqrt{x}}$; $\therefore \frac{\sqrt{a^2+x^2}}{\sqrt{x}}$ or $\frac{a^2+x^2}{x}$ is a minimum. Hence $2x^2\dot{x}-a^2\dot{x}-x^2\dot{x}=0$, and $x=a$, or the height $=$ the base.

### Prob. 8.

Given the length of an inclined plane; to find its height when the horizontal velocity of a body, after descending down its length, is a maximum.

Let $AB$ be the plane; produce $AB$ to $E$, and draw $ED$ perpendicular to the base $CB$ produced. Take $AB=a$, $AC=x$: then $BC=\sqrt{a^2-x^2}$. Now the velocity through $AB$ $=\sqrt{4mx}$, if $m=16\frac{1}{12}$; and this velocity : the horizontal velocity :: $BE : BD$



$:: BA : BC$

$:: a : \sqrt{a^2-x^2}$;

$\therefore$ the horizontal velocity $\propto \sqrt{x} \times \sqrt{a^2-x^2}$, a maximum. Hence $a^2x-x^3$ is a maximum, and $a^2\dot{x}-3x^2\dot{x}=0$, and $x=\frac{a}{\sqrt{3}}$, or $AB : AC :: \sqrt{3} : 1$.

If the base be given to find the height, it will be found in the same manner that $BC=CA$.

### Prob. 9.

Given the height of an inclined plane; required its length, so that a given power acting upon a given weight, in a direction parallel to the plane, may draw it up in the least time possible.

Let $a=$ the height, $x=$ the length, $P$ the power, and $W$ the weight. The tendency of $W$ down the plane is equal to

$\frac{W\times a}{x}$, and the accelerating force $=\frac{P-\frac{W\times a}{x}}{P+W}=\frac{Px-Wa}{\overline{P+W}\times x}$; but $T^2=\frac{S}{mF}$; $\therefore$ $T^2$ in this case $=\frac{x\times\overline{P+W}\times x}{m\times\overline{Px-Wa}}$; hence $\frac{x^2}{Px-Wa}$ is a minimum; $\therefore$ $2x\dot{x}\times\overline{Px-Wa}-Px^2\dot{x}=0$; and $Px=2aW$, or $P : W :: 2a : x ::$ twice the height : length.

### Prob. 10.

A body is projected from a given point, with a given velocity; to find the angle of elevation, when the horizontal range is a maximum.

Let $x=$ the versed sine of twice the angle of elevation; $2r=$ the parameter. The amplitude $\propto$ sine of twice the angle of elevation $\propto\sqrt{2rx-x^2}$; $\therefore$ by the Prob. $\sqrt{2rx-x^2}$, or $2rx-x^2$, is a maximum; hence $2r\dot{x}=2x\dot{x}$, and $x=r$, or the angle of projection is 45°.

### Prob. 11.

The same supposition being made, it is required to determine the angle of elevation, that the area of the parabola described may be a maximum.

The area of the parabola $=\frac{2}{3}\times$ base $\times$ altitude $\propto$ base $\times$ altitude $\propto$ sine $\times$ versed sine of twice the angle of elevation $\propto\sqrt{2rx-x^2}\times x$; $\therefore$ $\sqrt{2rx-x^2}\times x$, or its square $2rx^3-x^4$, is a maximum; hence $6rx^2\dot{x}-4x^3\dot{x}=0$, and $x=\frac{3r}{2}=$ versed sine of twice the angle of elevation.

### Prob. 12.

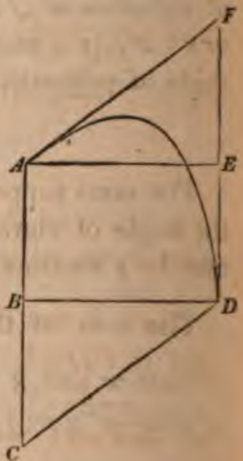
The same things being given, it is required to determine the angle of elevation, when the sum of the amplitude and altitude is a maximum.

In this case, since the altitude $= \frac{x}{4}$, we have $\frac{x}{4} + \sqrt{2rx - x^2}$ a maximum; $\therefore \frac{\dot{x}}{4} + \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - x^2}} = 0$; hence $\frac{\sqrt{2rx - x^2}}{4} = x - r$, and $x$ is found in a quadratic equation $= r \times \frac{\sqrt{17} \pm 1}{\sqrt{17}}$ = the versed sine of twice the angle of elevation.

### Prob. 13.

A body is projected with a given velocity from the top of a given tower; required the direction of projection, that it may fall at the greatest distance possible from the bottom.

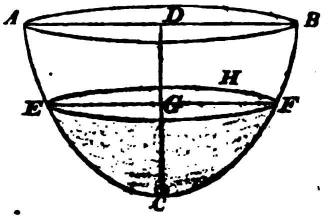
Let $AB$ the tower $= a$; suppose the body to be projected in the direction $AF$, and to fall at $D$. Let $DC$ parallel to $AF$ meet $AB$ produced in $C$, and draw $AE$ parallel to $BD$ meeting the perpendicular $DF$ in $E$. Take $FE$ or $BC = x$, and $p$ the parameter of the parabola. Then $p \times AC = CD^2$, or $p \times \overline{a+x} = CD^2$; and $DB^2 = DC^2 - CB^2 = p \times \overline{a+x} - x^2$; therefore $p \times \overline{a+x} - x^2$ is a maximum; and $p\dot{x} = 2x\dot{x}$, or $x = \frac{p}{2}$. Hence, in the right-angled triangle $AFE$, which is similar and equal to $DBC$, $AF$ and $FE$ are known, from which the angle of elevation $FAE$ may be found.



### Prob. 14.

To deduce a general expression for the time of emptying a vessel through an orifice in the bottom.

Let *AEB* represent the vessel, *C* the orifice, *CD* the perpendicular altitude, and *EHF* the descending surface of the fluid. Take $z$ to represent this surface, $n$ the orifice, $m = 32\frac{1}{6}$ feet, and $x = CG$, the height of the fluid at any point of its descent. Then the velocity of efflux is equal to that which a heavy body would acquire in falling through $\frac{x}{2}$; $\therefore$ the velocity of efflux $= \sqrt{mx}$. But by hydrostatical principles, as the area of the surface : area of the orifice :: the velocity at the orifice : the velocity of the descending surface; that is,



$$z : n :: \sqrt{mx} : \text{velocity at the surface} = \frac{n \times \sqrt{mx}}{z};$$

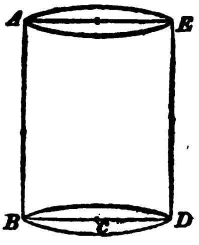
$\therefore \dot{T} = \frac{-z\dot{x}}{n\sqrt{mx}}$; the fluent of which, when corrected, gives the time required.

## Prob. 15.

1. To find the time in which a given cylinder will empty itself by an orifice in the base.

Here $z$ the descending surface is constant; and if $CD = b$, and $p = 3.14159$, $z = pb^2$; let $BA = a$; then $\dot{T} = \frac{-pb^2\dot{x}}{n\sqrt{mx}} = \frac{-pb^2x^{-\frac{1}{2}}\dot{x}}{n\sqrt{m}}$;

$\therefore T = \frac{pb^2}{n\sqrt{m}} \times -2x^{\frac{1}{2}} + \text{corr.} = \frac{2pb^2}{n\sqrt{m}} \times a^{\frac{1}{2}} - x^{\frac{1}{2}}$; and the whole time $= \frac{2pb^2a^{\frac{1}{2}}}{n\sqrt{m}}$.



In the same manner the time of emptying any prismatic vessel may be determined.

## PROB. 16.

2. To find the time of emptying a hemisphere by a hole in the vertex.

Let $ACB$ represent a hemisphere (See first Fig. in preceding page). Let $DA = a$, $CG = x$; then the descending surface $= p \times GF^2 = p \times \overline{2ax - x^2}$; and $\dot{T} = \frac{-p}{n\sqrt{m}} \times \frac{2ax\dot{x} - x^2\dot{x}}{x^{\frac{1}{2}}}$

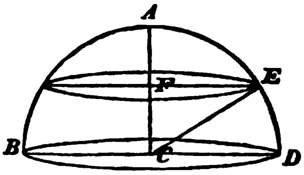
$= \frac{-p}{n\sqrt{m}} \times \overline{2ax^{\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x}}$; $\therefore\ T = \frac{-p}{n\sqrt{m}} \times \overline{\frac{4ax^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{5}} + \text{corr.}$

Let $x = a$, $T = 0$; $\therefore\ T = \frac{p}{n\sqrt{m}} \times \overline{\frac{4a^{\frac{5}{2}}}{3} - \frac{2a^{\frac{5}{2}}}{5}} - \frac{p}{n\sqrt{m}} \times \overline{\frac{4ax^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{5}}$. Let $x = 0$; then the whole time $= \frac{14p \times a^{\frac{5}{2}}}{15n\sqrt{m}}$.

## PROB. 17.

3. To find the same, the orifice being made in the base of the hemisphere.

Let $CF = x$; then $FE^2 = a^2 - x^2$; $\therefore\ p \times FE^2$ or $z = p \times \overline{a^2 - x^2}$; $\therefore\ \dot{T} = \frac{-p}{n\sqrt{m}} \times \frac{\overline{a^2\dot{x} - x^2\dot{x}}}{x^{\frac{1}{2}}}$



$= \frac{-p}{n\sqrt{m}} \times \overline{a^2x^{-\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x}}$; and $T =$

$\frac{-p}{n\sqrt{m}} \times \overline{2a^2x^{\frac{1}{2}} - \frac{2x^{\frac{5}{2}}}{5}} + C$; that is,

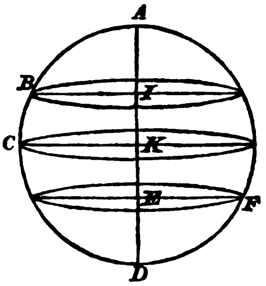
$T = \frac{p}{n\sqrt{m}} \times : 2a^{\frac{5}{2}} - \frac{2a^{\frac{5}{2}}}{5} - \overline{2a^2x^{\frac{1}{2}} - \frac{2x^{\frac{5}{2}}}{5}}$; $\therefore$ the whole time $=$ $\frac{8pa^{\frac{5}{2}}}{5n\sqrt{m}}$.

COR. If equal hemispheres are emptied by orifices in the vertex and the base, the time in the first case : the time in the last :: $\frac{14}{15} : \frac{8}{5}$

:: 7 : 12.

### PROB. 18.

4. A sphere *ABF* being filled with a fluid, it is required to compare the times of emptying the upper and lower hemispheres by an orifice at the bottom *D*.



Let $DI=x$; then, if $KA=a$, $K$ being the center, $z = p \times \overline{2ax-x^2}$; therefore $\dot{T}=\frac{-p}{n\sqrt{m}} \times \overline{2ax^{\frac{1}{2}}\dot{x}-x^{\frac{3}{2}}\dot{x}}$, and $T=\frac{-p}{n\sqrt{m}} \times \overline{\frac{4ax^{\frac{3}{2}}}{3}-\frac{2x^{\frac{5}{2}}}{5}}$ + corr. Now if $T=0$, $x=2a$; $\therefore T= \frac{p}{n\sqrt{m}} \times : \overline{\frac{4\times a\times \overline{2a}|^{\frac{3}{2}}}{3}-\frac{2\times \overline{2a}|^{\frac{5}{2}}}{5}} - \frac{p}{n\sqrt{m}} \times \overline{\frac{4ax^{\frac{3}{2}}}{3}-\frac{2x^{\frac{5}{2}}}{5}}$; or the whole time of emptying $= \frac{pa^{\frac{5}{2}}}{n\sqrt{m}} \times \frac{16\times\sqrt{2}}{15}$, since $x$ vanishes.

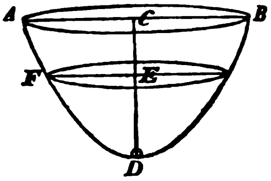
Again, if $x=a$, we get the time of emptying the lower hemisphere $=\frac{pa^{\frac{5}{2}}}{n\sqrt{m}} \times \frac{14}{15}$. Hence the time of emptying the upper part $=\frac{16\sqrt{2}-14}{15} \times \frac{pa^{\frac{5}{2}}}{n\sqrt{m}}$; $\therefore$ the time of emptying the upper part : time of emptying the lower :: $\frac{16\sqrt{2}-14}{15} : \frac{14}{15}$

:: $8\sqrt{2}-7 : 7$.

### PROB. 19.

5. Let the vessel be a paraboloid, and the orifice in the vertex.

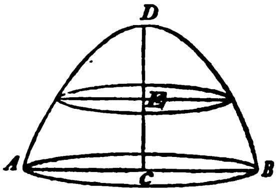
Let $c =$ the latus rectum; then, if $DE = x$, $FE^2 = cx$, and $z = pcx$; $\therefore \dot{T} = \frac{-pc}{n\sqrt{m}} \times \frac{x\dot{x}}{x^{\frac{1}{2}}} = \frac{-pc}{n\sqrt{m}} \times x^{\frac{1}{2}}\dot{x}$; $\therefore T = \frac{-pc}{n\sqrt{m}} \times \frac{2x^{\frac{3}{2}}}{3}$ + corr. Let $x = DC = a$; then $T = 0$; $\therefore$ the whole time $= \frac{pc}{n\sqrt{m}} \times \frac{2a^{\frac{3}{2}}}{3}$.



### PROB. 20.

6. Let the vessel be a paraboloid, and the orifice in the base.

Let $CE = x$, $DE = a - x$; $\therefore pc \times \overline{a-x} = z$; $\therefore \dot{T} = \frac{-pc}{n\sqrt{m}} \times \frac{\overline{a\dot{x} - x\dot{x}}}{x^{\frac{1}{2}}} = \frac{-pc}{n\sqrt{m}} \times \overline{ax^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x}}$, and $T = \frac{-pc}{n\sqrt{m}} \times \overline{2ax^{\frac{1}{2}} - \frac{2x^{\frac{3}{2}}}{3}}$ + corr. Now if $x = a$, $T = 0$; hence the whole time of emptying $= \frac{pc}{n\sqrt{m}} \times \frac{4a^{\frac{3}{2}}}{3}$.



COR. If the paraboloids in the last Examples are equal, the time of emptying the former : time of emptying the latter

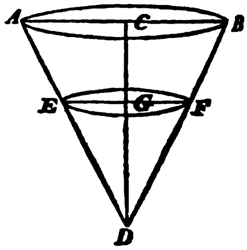
$:: \frac{2}{3} : \frac{4}{3}$

$:: 1 : 2$.

### PROB. 21.

7. Let $ADB$ be a cone, and the orifice in the vertex.

Let $DC = a$, $CB = b$, $DG = x$, $GF = y$; then $y = \frac{bx}{a}$, and $z$ or $py^2 = \frac{pb^2x^2}{a^2}$; $\therefore \dot{T} = \frac{-pb^2}{a^2n\sqrt{m}} \times x^{\frac{3}{2}}\dot{x}$; and the

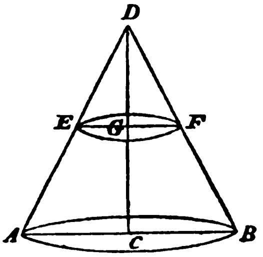


whole time of emptying $= \frac{pb^2}{a^2n\sqrt{m}} \times \frac{2a^{\frac{5}{2}}}{5} = \frac{2pb^2a^{\frac{1}{2}}}{5n\sqrt{m}}$.

**Cor.** The time of emptying a cone at the vertex : time of emptying a cylinder of the same base and altitude :: $\frac{2}{5}$ : 2 :: 1 : 5. (See Prob. 15.)

### Prob. 22.

8. To find the time when the orifice is in the base.



Here let $CG = x$; $\therefore\ DG = a - x$;

then $z$ or $p \times y^2 = \frac{pb^2}{a^2} \times \overline{a^2 - 2ax + x^2}$;

$\therefore\ \dot{T} = -\frac{pb^2}{a^2n\sqrt{m}} \times \overline{a^2x^{-\frac{1}{2}}\dot{x} - 2ax^{\frac{1}{2}}\dot{x} + x^{\frac{3}{2}}\dot{x}}$;

and the whole time $= \frac{16}{15} \times \frac{pb^2a^{\frac{1}{2}}}{n\sqrt{m}}$.

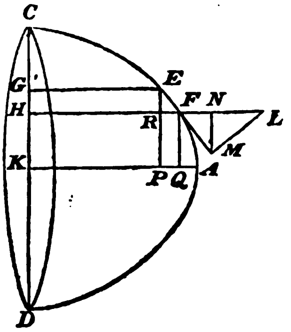
**Cor,** If the cones be equal in the two last cases, the time of emptying the former : time of emptying the latter

$:: \frac{2}{5} : \frac{16}{15}$

:: 3 : 8.

### Prob. 23.

Let *CAD* represent a plane figure or a solid, generated by the revolution of *CAD* about its axis *AK*. It is required to compare the resistance of the curve line *CAD*, and of the surface of the solid, with the resistance on the base *CD*; the plane figure or solid being supposed to move in a fluid in the direction of its axis.



Take *FE* a small arc; draw *FH*, *EG*, perpendicular to *CD*, and *FQ*, *EP*, perpendicular to *AK*. Let $AQ = x$, $QF = y$, $AF = z$; then ultimately $FE = \dot{z}$, $FR = \dot{x}$, and $ER = \dot{y}$. Let Let *LF* represent the force of one particle of water; draw *LM* perpendicular to a tangent *FM*, and *MN* to

*LF*; then, if this particle struck the base perpendicularly at *H*, it would be wholly effective; but its force on the curve at *F* is diminished in the ratio of *LF* to *LN*, or of $LF^2 : LM^2$; that is, of $FE^2 : ER^2$, or of $\dot z^2 : \dot y^2$.

Now let *CAD* be considered first as a plane surface. The number of particles which strike upon *FE* and *HG* is the same, and it varies as *GH* or $\dot y$. But the whole effect is as the number of particles × the force of each; ∴ the effect or the resistance on *FE* : resistance on *GH* :: $\frac{\dot y \dot y^2}{\dot z^2} : \dot y :: \frac{\dot y^3}{\dot z^2} : \dot y :: \frac{\dot y^3}{\dot x^2 + \dot y^2} : \dot y$, or :: $\frac{\dot y}{1 + \frac{\dot x^2}{\dot y^2}} : \dot y$; ∴ the whole resistance on the curve *CAD* : that on the base *CD* :: $f.\frac{\dot y}{1 + \frac{\dot x^2}{\dot y^2}} : f.\dot y$.

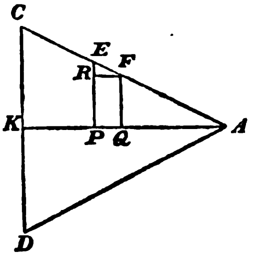
In the next place, suppose the figure to be a solid, generated by the revolution of *CAD* about its axis. In this case, the number of particles striking on the annulus, formed by the revolution of the part *GH*, is proportional to that annulus, or to the product of *GH* and the circumference described by *H*; that is, it varies as $2py\dot y$, or as $y\dot y$. Hence, since the resistance is proportional to the number of particles × the force of each, the resistance on the surface : that on the base :: $f.\frac{y\dot y^3}{\dot z^2} : f.y\dot y$

$$:: f.\frac{y\dot y^3}{\dot y^2 + \dot x^2} : f.y\dot y :: \frac{f.y\dot y}{1 + \frac{\dot x^2}{\dot y^2}} : f.y\dot y.$$

### PROB. 24.

1. To compare the resistance upon the sides of an isosceles triangle or prismatic solid *CAD*, moving in the direction of a perpendicular *AK*, with that on the base.

Here if $AC=s$, $CK=b$, $FQ=y$, $FE=\dot{z}$, $ER=\dot{y}$, $\dot{z}^2 : \dot{y}^2 :: s^2 : b^2$; $\therefore$ since the resistance on the sides : that on the base $:: f.\frac{\dot{y}\dot{y}^2}{\dot{z}^2} : f.\dot{y}$ (Prob. 23), it is $:: f.\frac{\dot{y}b^2}{s^2} : f.\dot{y} :: b^2 : s^2 :: KC^2 : CA^2$.



**Cor.** If $CAD$ be a right angle, and therefore $ACK=45°$, the resistance on the sides : that on the base :: 1 : 2.

### Prob. 25.

2. Let $CAD$ be a semi-circle moving in the direction $KA$.

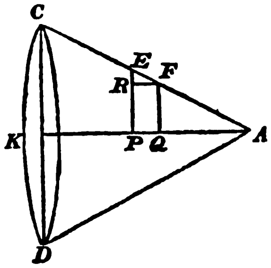
Here $\dot{z}=\frac{a\dot{y}}{\sqrt{a^2-y^2}}$ (Art. 44.); $\therefore$ $\dot{z}^2=\frac{a^2\dot{y}^2}{a^2-y^2}$. Now the resistance on $CAD$ : the resistance on $CD :: f.\frac{\dot{y}^3}{\dot{z}^2} : f.\dot{y}$; $\therefore$ in this case, they are $:: f.\frac{\dot{y}^3\times\overline{a^2-y^2}}{a^2\dot{y}^2} : f.\dot{y} :: f.\frac{a^2\dot{y}-y^2\dot{y}}{a^2} : f.\dot{y} :: y-\frac{y^3}{3a^2} : y ::$ (if $y=a$) $a-\frac{a}{3} : a :: 2 : 3$.

### Prob. 26.

3. Let $CAD$ represent a cone.

Here if $CK=b$, and $AC=s$, $\frac{\dot{y}^2}{\dot{z}^2}=\frac{b^2}{s^2}$; $\therefore$ since res. on surface : res. on base $:: f.\frac{y\times\dot{y}^3}{\dot{z}^2} : f.y\dot{y}$, we have, res. on surface : res. on base $:: f.y\dot{y}\times\frac{b^2}{s^2} : f.y\dot{y} :: b^2 : s^2$

$:: CK^2 : CA^2$.



**Cor. 1.** If $CAD$ be a right angle, the former resistance : the latter :: 1 : 2.

Cor. 2. If a right-angled cone and cylinder, whose bases are the same, move in the same fluid in the direction of their axes, and the resistance upon the cone = resistance on the cylinder; then the velocity of the cone : velocity of the cylinder :: $\sqrt{2} : 1$. For let $V$ represent the velocity of the cone, and $v$ of the cylinder; then, since by the last Cor. when the velocities are the same, the resistances are as $b^2 : s^2$, and that, cæteris paribus, the resistance varies as the square of the velocity; ∴ in this case, resistance on cone : resistance on the cylinder :: $V^2b^2 : v^2s^2$; whence $V^2b^2 = V^2s^2$, and $V : v :: s : b :: \sqrt{2} : 1$.

## Prob. 27.

4. To compare the resistances upon a globe and cylinder of equal diameters, moving with equal velocities in the direction of the axis of the cylinder.

In general, the fluxion of the resistance on the solid : the fluxion of that on the base :: $\frac{y\dot{y}^3}{\dot{z}^2} : y\dot{y}$, since $\dot{z}^2 = \frac{a^2\dot{y}^2}{a^2-y^2}$, (Art. 44.) :: $\frac{\overline{a^2-y^2}\times y\dot{y}}{a^2} : y\dot{y}$,

:: $y\dot{y} - \frac{y^3\dot{y}}{a^2} : y\dot{y}$; ∴ the whole resistance on the globe : that on the cylinder :: $\frac{y^2}{2} - \frac{y^4}{4a^2} : \frac{y^2}{2}$; if $y=a$, :: $\frac{2a^2-a^2}{4} : \frac{a^2}{2} :: 1 : 2$.

## Prob. 28.

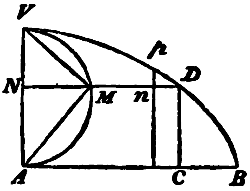
5. Let the solid be the common paraboloid moving in the direction of its axis.

Here $y^2=ax$; ∴ $\dot{x} = \frac{2y\dot{y}}{a}$, and $\frac{\dot{x}^2}{\dot{y}^2} = \frac{4y^2}{a^2}$; hence, since resistance on solid : resistance on the base :: $f.\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} : f.y\dot{y}$, they are $:: f \cdot \dfrac{y\dot{y}}{1+\dfrac{4y^2}{a^2}} : f.\, y\dot{y} :: \dfrac{a^2}{8} \times \text{hyp.log.}\overline{\dfrac{a^2}{4}+y^2} : \dfrac{y^2}{2}$

$$\text{(Art. 42. Ex. 5.)} :: \frac{a^2}{4} \times \text{hyp.log.}\,\overline{\frac{a^2}{4}+y^2} : y^2.$$

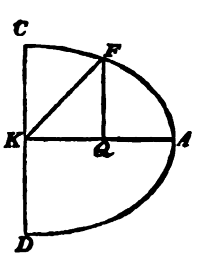
## Prob. 29.

6. A solid, generated by the revolution of the common cycloid about its base, moves in a fluid in the direction of its base; to compare the resistance against this solid with that against its circumscribing cylinder.



Let $VA=2a$, $BC=x$, $CD=y$; then $\dot{x} : \dot{y} :: Dn : np :: MN : NV :: \sqrt{AN} : \sqrt{NV}$, or $\dot{x}^2 : \dot{y}^2 :: y : 2a-y$; $\therefore \dfrac{\dot{x}^2}{\dot{y}^2} = \dfrac{y}{2a-y}$; hence resistance on solid : resistance on the cylinder $:: f.\, \dfrac{y\dot{y}}{1+\dfrac{y}{2a-y}} : f.\, y\dot{y} ::$ $f.\, \dfrac{2ay\dot{y}-y^2\dot{y}}{2a} : f.\, y\dot{y} :: \dfrac{y^2}{2} - \dfrac{y^3}{6a} : \dfrac{y^2}{2}$; that is, if $y=2a$, resistance on the solid : resistance on the cylinder $:: 2a^2 - \dfrac{4a^2}{3} : 2a^2 :: 1 : 3$.

## Prob. 30.



7. Let $CAD$ be a spheroid, whose center is $K$, moving in the direction of the major axis $KA$; to compare the resistances as before.

Let $AQ=x$, $KQ=u$, $QF=y$; then, if $AK=a$, and the latus rectum $= 2r$, by a property of conic sec[illegible], $a^2-u^2$ :

$y^2 :: a : r$; $\therefore\ a^2 - u^2 = \frac{ay^2}{r}$, and $u = \sqrt{a^2 - \frac{ay^2}{r}}$, and $\dot{u} = \frac{-ay\dot{y}}{r\sqrt{a^2 - \frac{ay^2}{r}}}$; but since $u + x = a$, $\dot{u} = -\dot{x}$; $\therefore\ \dot{x} = \frac{ay\dot{y}}{r \times \sqrt{a^2 - \frac{ay^2}{r}}}$, and $\frac{\dot{x}^2}{\dot{y}^2} = \frac{a^2y^2}{a^2r^2 - ary^2} = \frac{ay^2}{ar^2 - ry^2}$, and $1 + \frac{\dot{x}^2}{\dot{y}^2} = 1 + \frac{ay^2}{ar^2 - ry^2} = \frac{ar^2 + \overline{a - r}\,.\,y^2}{ar^2 - ry^2}$; therefore $\frac{y\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}} = \frac{ar^2 - ry^2}{ar^2 + \overline{a - r}\,.\,y^2} \times y\dot{y}$, whose fluent (by Fluent 10.) $= \frac{-ry^2}{2 \times \overline{a - r}}$ $+ \frac{\frac{1}{2}a^2r^2}{\overline{a - r}|^2} \times$ hyp. log. $\overline{y^2 + \frac{ar^2}{a - r}} = A$; $\therefore$ resistance against the spheroid : resistance on the cylinder $:: A : \frac{y^2}{2}$

$:: 2A : y^2$.

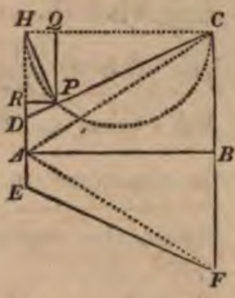
### PROB. 31.

8. If the figure be an hyperboloid, by conic sections, $u^2 - a^2 : y^2 :: a : r$; and in this case, $\frac{y\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}} = \frac{\overline{r^2a + ry^2} \times y\dot{y}}{r^2a + \overline{a + r}\,.\,y^2}$, whose fluent $= \frac{\frac{1}{2}ry^2}{r + a} + \frac{\frac{1}{2}r^2a^2}{\overline{r + a}|^2} \times$ hyp. log. $\overline{y^2 + \frac{ra^2}{r + a}} = B$ (Fluent. 10.); $\therefore$ resistance against the solid : resistance on the cylinder $:: 2B : y^2$.

### PROB. 32.

9. To determine the frustum *CDEF* of a triangular prism, of a given base *CF*, and altitude *BA*, which, moving in a medium in the direction of its length *BA*, shall be resisted the least [illegible] possible.

Let $CH$ drawn parallel to $AB$ meet $ED$ produced in $H$; on $HC$ describe a semi-circle, meeting $DC$ in $P$; join $HP$, and draw $PQ$ perpendicular to $HC$, and $PR$ to $DH$. Then the resistance to $DC$, which varies as the number of particles × the force of each, varies as $\frac{DH \times DH^2}{DC^2}$, or as $\frac{DH \times DR^2}{DP^2}$; but $DH : DP :: DP : DR$, since the angle $DPH$ is a right angle; $\therefore DH : DR :: DH^2 : DP^2$, or $:: DP^2 : DR^2$; $\therefore DR = \frac{DH \times DR^2}{DP^2}$, or the resistance to $DC$ varies as $DR$; and the resistance to $AD$ and $DC$ varies as $AD + DR$. Now this is a minimum when $DR$ is a minimum, or when $RH$ is a maximum; but $RH = PQ$, and $PQ$ is a maximum (Art. 23. Ex. 2.) when $CQ = QH$; that is, when $DCH$ or $DCB = 45°$.

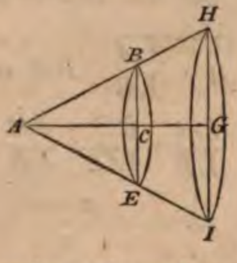


Cor. In this case $BC$ is supposed to be greater than $BA$; if not, the whole prism will be less resisted than any frustum $CDEF$ of a greater prism.

### Prob. 33.

10. To determine that frustum of a cone of a given base and altitude, which, moving in the direction of its axis, shall be less resisted than any other.

Let $HBEI$ be the required frustum, and $HAI$ the complete cone. Then, by Prob. 26, the resistance on the surface $AH$ : that on the base $HI :: HG^2 : AH^2$; $\therefore$ if the resistance on the base $= 1$, that on the surface $= \frac{HG^2}{AH^2}$.



Also, since the resistance varies as the number of particles into the force of each, the resistance on the base $HI$ : that on the base $BE$ :: $HG^2 : BC^2$, the force of each being the same when the velocity is the same; ∴ the resistance on the base $BE = \frac{BC^2}{HG^2} = \frac{AB^2}{AH^2}$; and the resistance on the base $BE$ : that on the surface $BAE$ :: $AB^2 : BC^2$; therefore the resistance on the surface $BAE = \frac{BC^2}{HG^2} \times \frac{HG^2}{AH^2} = \frac{BC^2}{AH^2}$. Hence the resistance on the surface of the frustum $BHIE = \frac{HG^2 - BC^2}{AH^2}$; and the whole resistance $= \frac{AB^2 + HG^2 - BC^2}{AH^2}$ $= \frac{AC^2 + HG^2}{AH^2}$.

Let $AG = x$, $CG = a$, $HG = b$; then $AH^2 = b^2 + x^2$; and the resistance $= \frac{\overline{x-a}|^2 + b^2}{x^2 + b^2} = \frac{x^2 - 2ax + a^2 + b^2}{x^2 + b^2} = 1 + \frac{a^2 - 2ax}{x^2 + b^2}$, which is a minimum; ∴ $\frac{-2a\dot{x} \times \overline{x^2 + b^2} + 2x\dot{x} \times \overline{2ax - a^2}}{\overline{x^2 + b^2}|^2}$ $= 0$; hence $x^2 + b^2 = 2x^2 - ax$; ∴ $x^2 - ax = b^2$, and $x = \frac{a}{2}$ $\pm \sqrt{\frac{a^2}{4} + b^2}$.

Cor. The Thirty second Problem might be solved by the same process; the only difference is, that the resistance on the base $CF$ in that Problem : that on $DE$ :: $CB : DA$; and the expression, which is a minimum, $= \frac{LA}{LB} + \frac{BC^2}{LC^2} \times \overline{1 - \frac{LA}{LB}}$, $BA$ and $CD$ being produced to meet in $L$, or $\frac{LA}{LB} + \frac{BC^2}{LC^2} \times \frac{AB}{LB}$; which, according to the same substitution, varies as $\frac{x-a}{x} + \frac{b^2 a}{x^3 + b^2 x}$; and the fluxion of this quantity being put $= 0$, gives $x = b$, or the angle $DCB = 45°$, as before.

### PROB. 34.

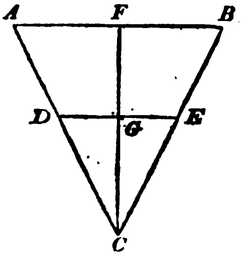
To find at what angle the wind must act upon the sails of a mill, so that its effect to produce motion may be a maximum.

Let $x$ = the cosine of the angle; then if $r$ = radius, the sine $=\sqrt{r^2-x^2}$; but the effect by hydrostatics varies as $x\times\overline{r^2-x^2}$; or $r^2x-x^3$ is a maximum; hence $r^2\dot{x}-3x^2\dot{x}=0$, and $x=\dfrac{r}{\sqrt{3}}$, the cosine of 54°. 44′.

### PROB. 35.

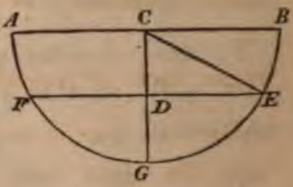
Let the triangle *ABC* be immersed in a fluid, so that its base may be level with the surface; to find where a line *DE* must be drawn parallel to the base, so that the pressure upon it may be a maximum.

Draw *CF* perpendicular to *AB*; let $AB=a$, $CF=b$, $FG=x$; then $CF : CG :: AB : DE$, or $b : b-x :: a : DE$, $=a\times\dfrac{\overline{b-x}}{b}$. Now the pressure varies as the surface pressed into the depth of the center of gravity; ∴ the pressure in this case varies as $\overline{b-x}\times x\propto bx-x^2$, a maximum; hence $b\dot{x}-2x\dot{x}=0$, and $x=\dfrac{b}{2}$.



### PROB. 36.

If a semicircle *ACB* be immersed vertically in a fluid, with its diameter contiguous to the surface; to find on which of the chords parallel to *AB* there is the greatest pressure, the density of the fluid being supposed to vary as the depth, Let $CD = x$, $CE = r$; then the pressure on $FE$ varies as $FE \times$ depth $\times$ density $\propto 2\sqrt{r^2 - x^2} \times x^2$, a maximum; hence the fluxion of $r^2x^4 - x^6 = 0$, or $4r^2x^3\dot{x} = 6x^5\dot{x}$; $\therefore$ $x^2 = \frac{4r^2}{6} = \frac{2r^2}{3}$, and $x = r \times \sqrt{\frac{2}{3}}$.



If $AGB$ be a parabola, on the same supposition, $x = \frac{4b}{5}$, $b$ being assumed $= CG$.

### PROB. 37.

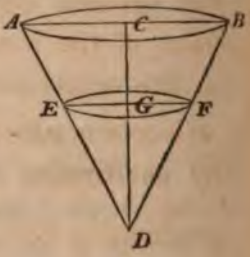
Let $AGB$ represent a hemisphere; to find the section $FE$ parallel to the surface, on which the pressure is a maximum, the density of the fluid being as the $n^{th}$ power of the depth.

The area of the section $FE$ varies as $y^2$, or as $\overline{r^2 - x^2}$; hence the pressure varies as $\overline{r^2 - x^2} \times x \times x^n$; that is, $r^2x^{n+1} - x^{n+3}$ is a maximum; hence $\overline{n+1} \,.\, r^2x^n\dot{x} = \overline{n+3} \,.\, x^{n+2}\dot{x}$, and $x = \sqrt{\frac{n+1}{n+3}} \times r$.

### PROB. 38.

A cubic inch of metal, whose specific gravity is to that of water :: $m : 1$, is formed into a hollow cone, and immersed with its vertex downward; it is required to find the ratio of the exterior diameter of its base to the altitude, when the surface immersed is a minimum.

Let $r =$ the radius $BC$ of the base, $DB$ the altitude $= x$, $DG$ the altitude to which it is immersed $= z$, $p = 3.14159$, &c. Then, by similar triangles, $x : r :: z : \frac{rz}{x} = GF$; and the content of the conical part $EDF = p \times GF^2 \times \frac{GD}{3} = \frac{pr^2z^3}{3x^2} =$ the bulk of water displaced; and the weight of the



water displaced, or $\frac{pr^2z^3}{3x^2} \times 1$, $= m \times 1$, the weight of the cubic inch of metal; $\therefore\ z = \sqrt[3]{\frac{3m}{pr^2}} \times x^{\frac{2}{3}}$. Also, since the surface of a cone $= \frac{1}{2}$ the circumference of the base multiplied into the slant side, the surface of $ADC = pr \times \sqrt{r^2+x^2}$; but similar surfaces are as the squares of corresponding sides; $\therefore\ x^2 : z^2 :: pr \times \sqrt{r^2+x^2} : \frac{prz^2 \times \sqrt{r^2+x^2}}{x^2}$, the surface immersed; which, by substitution, $= \frac{pr \times \sqrt{r^2+x^2}}{x^2} \times \overline{\frac{3m}{pr^2}}\,^{\frac{2}{3}} \times x^{\frac{4}{3}}$, a minimum by the Problem; or $\frac{\sqrt{r^2+x^2}}{x^{\frac{2}{3}}}$ is a minimum; and the fluxion being put $= 0$, $x = r \times \sqrt{2}$.

### Prob. 39.

A cylinder of oak is immersed in water till its top is just level with the surface, and then is suffered to ascend; it is required to determine the greatest altitude to which it will rise, the velocity which it has then acquired, and the time of its ascent.

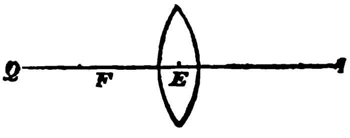
Let $h =$ the height, and $a$ the base of the cylinder, and suppose the specific gravity of oak : that of water $:: n : 1$. Let $x$ be any variable altitude through which the cylinder has ascended, and $l = 16\frac{1}{12}$ feet. Then the moving force by which the cylinder endeavours to descend $= h \times a \times n$, and the force of the water upwards to prevent it $= \overline{h-x} \times a \times 1$; $\therefore$ the whole moving force upon the cylinder $= \overline{h-x} \times a - h \times a \times n = ah - ax - h \times a \times n = \overline{1-n}\,.\,ha - ax = mha - ax$, by substituting $m$ for $1 - n$, $= a \times \overline{mh - x}$. Hence the accelerating force $= \frac{a \times \overline{mh-x}}{h \times a \times n} = \frac{mh-x}{nh}$. Now if $v$ represent the velocity of the cylinder after it has risen through a space

$=x$, $v\dot{v} = \pm\, 2lF\dot{x} =$, in this case, $2l \times \frac{mh\dot{x} - x\dot{x}}{nh}$; $\therefore\ v^2 = 2l \times \frac{2mhx - x^2}{nh}$, and $v = \sqrt{\frac{2l}{nh}} \times \sqrt{2mhx - x^2}$. And when the cylinder has acquired its greatest ascent, $v = 0$, or $\sqrt{2mhx - x^2} = 0$; $\therefore\ x = 2mh =$ the part of the cylinder extant.

To find the time, we have $\dot{T} = \frac{\dot{x}}{v} = \sqrt{\frac{nh}{2l}} \times \frac{\dot{x}}{\sqrt{2mhx - x^2}} = \sqrt{\frac{nh}{2lm^2h^2}} \times \frac{mh\dot{x}}{\sqrt{2mhx - x^2}}$, and $T = \sqrt{\frac{n}{2lm^2h}} \times A$; where $A =$ a circular arc of radius $mh$, and versed sine $x$, which needs no correction.

## PROB. 40.

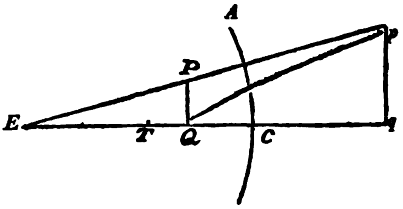
Let $Q$ be an object placed beyond the principal focus $F$ of a convex lens; to find its position when its distance $Qq$ from its image $q$ is a minimum.



Let $QE = x$, $FE = a$; then $QF : QE :: QE : Qq$, or $x - a : x :: x : Qq = \frac{x^2}{x - a}$, which is a minimum; $\therefore$ $2x\dot{x} \times \overline{x - a} - x^2\dot{x} = 0$; that is, $2x^2\dot{x} - 2ax\dot{x} - x^2\dot{x} = 0$; $\therefore$ $x = 2a$, or $QE = 2EF$, and $QF = FE$.

## PROB. 41.

If a person view himself in a concave reflector, his image decreases from the reflector to the principal focus, and then increases in going from it.

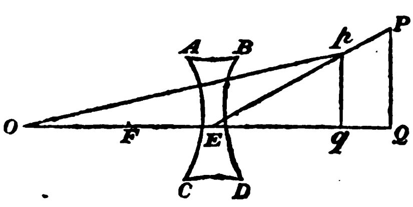


Let $E$ be the center, and $T$ the principal focus of the concave reflector $AC$. Let $pq$ be the image of $PQ$; the apparent magnitude of $PQ$ to an eye situated at $Q$ is

$\frac{pq}{Qq}$. Take $ET=TC=a$, $QC=x$, $PQ=M$; then $TQ : TE :: TE : Tq=\frac{TE^2}{TQ} = \frac{a^2}{a-x}$; $\therefore Cq=Tq-TC=\frac{a^2}{a-x} - a=\frac{ax}{a-x}$, and $Qq=x+\frac{ax}{a-x}=\frac{2ax-x^2}{a-x}$. Also $(M)$ $PQ : pq :: QC : qC :: x : \frac{ax}{a-x} :: a-x : a$; $\therefore pq = \frac{Ma}{a-x}$; $\therefore$ the apparent magnitude or $\frac{pq}{Qq} \propto \frac{Ma}{a-x} \times \frac{a-x}{2ax-x^2} \propto \frac{1}{2ax-x^2}$, whose fluxion $= \frac{2x\dot{x}-2a\dot{x}}{\overline{2ax-x^2}^{\,2}} = \frac{2\dot{x} \times \overline{x-a}}{\overline{2ax-x^2}^{\,2}}$, which is negative, while $x$ is less than $a$, or $QC$ less than $TC$; hence the apparent magnitude is at that time decreasing. But when $x$ is greater than $a$, or $QC$ greater than $TC$, the expression for the fluxion of the apparent magnitude becomes positive, and the magnitude of the image increases.

## Prob. 42.

If the eye and an object be both fixed, and a concave lens be moved from the object toward the eye, the apparent magnitude of the object will decrease to the middle point, and then begin to increase. Required a proof.



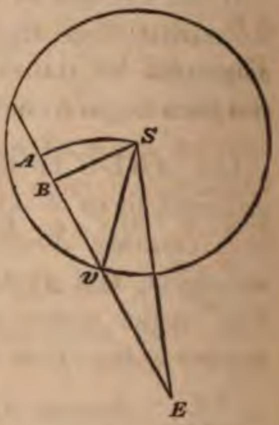
Let $ABDC$ be the lens, $O$ the place of the eye, $F$ the principal focus of rays coming in a contrary direction, $PQ$ the object, and $pq$ the image. The angle, under which $pq$ appears to the eye at $O$, varies as $\frac{pq}{qO}$. Assume $QO = 2a$, $FE=p$, $QE=x$, and $PQ=M$.

Then, since $QF : FE :: QE : Eq$, $Eq = \frac{QE \times FE}{QF} = \frac{px}{p+x}$. Also $QF : QE :: QE : Qq$; $\therefore$ $Qq = \frac{QE^2}{QF}$

$=\frac{x^2}{p+x}$; hence $Oq=OQ-Qq=2a-\frac{x^2}{p+x}=\frac{2ap+2ax-x^2}{p+x}$. And $PQ$ $(M)$ : $pq$ :: $QE$ : $qE$ :: $x$ : $\frac{px}{p+x}$; $\therefore pq=\frac{M\times p}{p+x}$. Hence the apparent magnitude, or the angle $qop$, which varies as $\frac{pq}{qO}$, varies as $\frac{Mp}{p+x}\times\frac{\overline{p+x}}{2ap+2ax-x^2}$, $\propto \frac{1}{2ap+2ax-x^2}$. The fluxion of this quantity is $\frac{2\dot{x}\times\overline{x-a}}{\overline{2ap+2ax-x^2}|^2}$, which is negative, while $x$ is less than $a$, but positive, when $x$ becomes greater; that is, the apparent magnitude decreases to the middle point, and afterwards increases.

### PROB. 43.

To find the position of Venus when brightest.

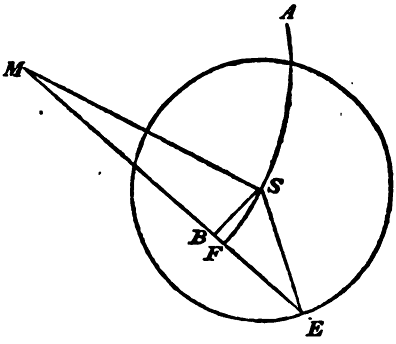


Let $E$ be the Earth, $S$ the Sun, and $V$ Venus; join $SV$, $SE$, $EV$, and produce $EV$ to $A$, making $VA=VS$; with $V$ as a center and $VS$ radius describe the circular arc $SA$, and draw $SB$ perpendicular to $EA$. Then $SEV$ is the angle of elongation, $SVA$ the exterior angle, $VB$ its cosine, and $BA$ its versed sine to the radius $SV$. Take $SE=a$, $EV=x$, $VB=y$, $SV=b$. Then the visible illumined part $\propto BA \propto b-y$; and the brightness $\propto\frac{b-y}{x^2}$, $\propto\frac{b}{x^2}-\frac{y}{x^2}$, which by the Problem is a maximum. Now $SE^2=EV^2+SV^2+2EV\times VB$, or $a^2=x^2+b^2+2x\times y$; $\therefore y=\frac{a^2-b^2-x^2}{2x}=$ (if $m^2=a^2-b^2$) $\frac{m^2-x^2}{2x}$; $\therefore$ by substituting for $y$, we have

$\frac{b}{x^2} - \frac{m^2 - x^2}{2x^3}$ a maximum; that is, $\frac{2bx - m^2 + x^2}{2x^3}$ is a maximum; hence $\overline{2b\dot{x} + 2x\dot{x}} \times 2x^3 - 6x^2\dot{x} \times \overline{2bx - m^2 + x^2} = 0$; or if we divide by $2x^2\dot{x}$, it may be reduced to this form; $-x^2 - 4bx + 3m^2 = 0$; $\therefore x^2 + 4bx = 3m^2$, from which equation $x = -2b + \sqrt{4b^2 + 3m^2} = -2b + \sqrt{3a^2 + b^2}$. Hence the three sides of the triangle *SEV* are known, to find the angle of elongation *SEV*, which is equal to 39°. 44′.

## Prob. 44.

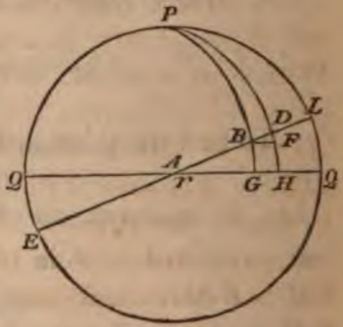
To find the position of Mars when least bright.



Let *E* represent the Earth, *S* the Sun, *M* Mars at the time required. Join *SE*, *SM*, *EM*; and suppose *EM* to be produced to *D*, making *MD* = *MS*. With *M* as a center and *MS* radius describe the circle *ASF*; draw *SB* perpendicular to *DE*. Then *SEM* is the angle of elongation, and *SMD* the exterior angle of elongation, whose versed sine is *DB*. Take $ES = a$, $MS = b$, $EM = x$, $MB = y$; then the brightness $\propto \frac{b+y}{x^2}$. But $ES^2 = EM^2 + MS^2 - 2EM \times MB$, or $a^2 = x^2 + b^2 - 2xy$; $\therefore y = \frac{b^2 - a^2 + x^2}{2x} = \frac{m^2 + x^2}{2x}$, if $m^2 = b^2 - a^2$; $\therefore$ the brightness $\propto \frac{b}{x^2} + \frac{m^2 + x^2}{2x^3} \propto \frac{2bx + m^2 + x^2}{2x^3}$, which is a minimum. And the fluxion $\overline{2b\dot{x} + 2x\dot{x}} \times 2x^3 - 6x^2\dot{x} \times \overline{2bx + m^2 + x^2} = 0$, from which equation $x$ may be found $= -2b + \sqrt{4b^2 - 3m^2} = -2b + \sqrt{b^2 + 3a^2}$.

## Prob. 45.

The Sun being supposed to move uniformly in the ecliptic; to find when that part of the equation of time, which arises from the obliquity of the ecliptic, is a maximum.



Let *QQ* represent the equinoctial, *EL* the ecliptic, *A* the first point of Aries, *PLQ* the solstitial colure. Then at *A* the Sun's longitude and right ascension are equal; and they are again equal when the Sun is at *L*; but at any intermediate point, as *B*, *PBG* being a declination circle, the longitude *AB* is greater than the right ascension *AG*; it appears, therefore, that from *A* to a certain point the longitude increases faster than the right ascension, and from that point to *L* the right ascension increases faster than the longitude; hence the equation is a maximum, when the daily increment of right ascension is equal to the daily increment of longitude. At that time let the Sun be at *B*, and let *BD* be the increment of longitude, and *GH* of right ascension, in one day. Draw the circle of declination *PDH*, and the small arc *BF* parallel to *GH*.

Now $GH : BF :: r : \cos. BG$,

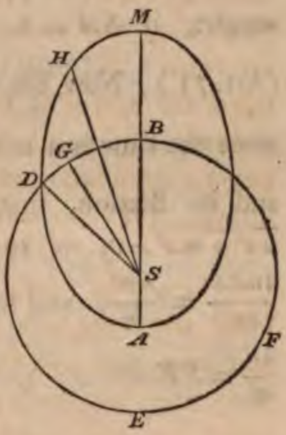
and $BF : BD :: S, BDF : r$ (since the triangle *BDF* is extremely small);

$\therefore\ GH : BD :: S, BDF : \cos. BG :: S, ABG : \cos. BG$, and $GH = BD$ by hypoth.; $\therefore\ S, ABG = \cos. BG$. But by Napier's Rules, $r \times \cos. \angle BAG = S, ABG \times \cos. BG$; that is, $r \times \cos. BAG = \overline{\cos.}\rceil^{2} BG$; hence $r : \cos. BG :: \cos. BG : \cos. BAG$; or the equation is a maximum, when the cosine of declination is a mean proportional between radius and the cosine of the obliquity of the ecliptic. The longitude at that time $= 46^\circ.\ 14'$.

## Prob. 46.

To find when that part of the equation of time is a maximum, which arises from the unequal angular motion of the Sun in the ecliptic.

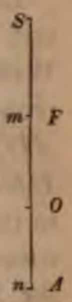
Let $MDA$ represent the ecliptic, $MSA$ its major axis, and $S$ the Earth in one of its foci. With $S$ as a center, and $SD$ a mean proportional between the semi-axes of $ADM$ as radius, describe the circle $DEF$. The area of this circle is equal to that of the ellipse; ∴ if a body be conceived to revolve in this circle with the Sun's mean angular velocity, its periodic time will equal that of the Sun in the ecliptic; for the areas described in the two cases *dato tempore* are the same. Let this imaginary body be conceived to set off from $B$ at the same time that the Sun begins its motion from the higher apse at $M$. The Sun's velocity at $M$ is less than the mean; therefore the angle $BSG$ described by the body in some given time is greater than $BSH$ described by the Sun in the same time; and their difference, or the equation, will continue to increase till the angular velocity of the Sun is equal to that of the body; hence the equation at that point is a maximum. Now the angular velocity varies as the area described in a given time directly, and the square of the distance inversely; ∴ since the area described in a given time is the same in both cases, when the angular velocities are equal, the distances are equal; or the equation is a maximum when the Sun is at $D$.



The absolute equation from both these causes is a maximum about the first of November.

### Prob. 47.

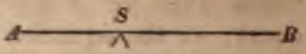
To a pendulum $SA$ of a given length, suspended at $S$, a given weight $n$ is affixed at $A$; to find where another weight must be fixed, so that it may vibrate in the least time possible.



Let $F$ be the point required, and $O$ the center of oscillation; then the pendulum itself being considered as of no weight, if $SA = a$, and $SF = x$, $SO = \frac{na^2 + mx^2}{na + mx}$ (Art. 71.) Now the time of oscillation $\propto \sqrt{SO}$; $\therefore$ since the time is a minimum, $\frac{na^2 + mx^2}{na + mx}$ is a minimum; and its fluxion, that is, $2mx\dot{x} \times \overline{na + mx} - m\dot{x} \times \overline{na^2 + mx^2} = 0$, or $2nax + 2mx^2 = na^2 + mx^2$; $\therefore x^2 + \frac{2nax}{m} = \frac{na^2}{m}$; and from this quadratic $x = \frac{a}{m} \times \sqrt{n^2 + mn} - \frac{na}{m} = SF$.

### Prob. 48.

Let $AB$ represent a straight lever moveable round an horizontal axis of motion, which passes through $S$; suppose a weight $q$ to be affixed to the extremity of the shorter arm $SA$, and a power $p$ at the extremity of the longer $SB$; Required the ratio of the arms, when the effect of $p$ to turn the system at the first instant of motion is a maximum, the inertia of the lever not being considered.



Let $SA = x$, $SB = a$. The moving force of $p = p$; but the weight $q$ would balance a power at $B = \frac{q \times AS}{BS} = \frac{qx}{a}$; $\therefore$ the moving force acting on $B = p - \frac{qx}{a} = \frac{pa - qx}{a}$.

Also the inertia or mass is obtained by supposing the bodies $p$ and $q$ to be removed, and a weight $=p+\frac{qx^2}{a^2}$, or $\frac{pa^2+qx^2}{a^2}$ placed at $B$ (Art. 65.); hence the force, which accelerates $B$ at the first instant of motion, $=\frac{\overline{pa-qx}\times a}{pa^2+qx^2}$, which is in this case a maximum;

$$\therefore\ -q\dot{x}\times\overline{pa^2+qx^2}-2qx\dot{x}\times\overline{pa-qx}=0,$$
$$\text{or } pa^2+qx^2+2pax-2qx^2=0;$$

and by solving the quadratic, $x=\frac{pa+\sqrt{p^2a^2+pqa^2}}{q}$.

## Prob. 49.

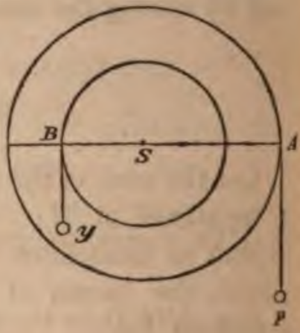
Let the arms of the lever $ASB$ be given, and a given power $p$ be affixed at $B$; required the weight $y$ suspended at $A$, so that its momentum in a small given time may be a maximum, the inertia of the lever not being considered. (See preceding Fig.)

Let $SB=a$, $SA=b$; then the moving force at $B$, as before, $=p-\frac{y\times b}{a}=\frac{pa-yb}{a}$; and the inertia $=p+y\times\frac{b^2}{a^2}=\frac{pa^2+yb^2}{a^2}$; $\therefore$ the accelerating force upon $B$ at first $=\frac{\overline{pa-yb}}{pa^2+yb^2}\times a$. Now the force which accelerates $B$ : that which accelerates $A$ :: $SB$ : $SA$ :: $a$ : $b$; $\therefore$ the force which accelerates $A=\frac{\overline{pa-yb}}{pa^2+yb^2}\times b$; and this varies as the velocity; hence, at the commencement of the motion, the momentum $\propto\frac{pay-by^2}{pa^2+b^2y}$, a maximum; $\therefore\ \overline{pa\dot{y}-2by\dot{y}}\times\overline{pa^2+b^2y}-b^2\dot{y}\times\overline{pay-by^2}=0$; hence, dividing by $\dot{y}$, and multiplying the quantities, $b^3y^2+2pa^2by=p^2a^3$, and $y=\frac{pa}{b^2}\times\sqrt{ab+a^2}-\frac{pa^2}{b^2}$.

### Prob. 50.

Given the radii $SA$, $SB$ of a wheel and axle, and let a given weight $p$, applied at the circumference of the wheel, raise a weight $y$ applied at the circumference of the axle; to find $y$ when the momentum communicated to it in a given time is a maximum, the inertia of the wheel and axle not being considered.

Let $SA=a$, $SB=b$; then the moving force upon $A=p-\frac{yb}{a}$; and the mass supposed to be collected at $A=p+\frac{yb^2}{a^2}$; therefore the accelerating force at $A=\frac{\overline{pa-yb}}{pa^2+yb^2}\times a$; hence the accelerating force at $B=\frac{\overline{pa-yb}}{pa^2+yb^2}\times b$; $\therefore$ the momentum communicated to $y \propto \frac{pay-y^2b}{pa^2+yb^2}$, a maximum; and the fluxion $=0$, or $\overline{pa\dot{y}-2by\dot{y}}\times\overline{pa^2+yb^2}-b^2\dot{y}\times\overline{pay-y^2b}=0$; whence, dividing by $\dot{y}$, and multiplying the quantities $b^3y^2+2pa^2by=p^2a^3$, the same equation as in the last case; and $y=\frac{pa}{b^2}\times\sqrt{ab+a^2}-\frac{pa^2}{b^2}$.



### Prob. 51.

Given two weights $p$ and $q$, acting, as in the former case, at $A$ and $B$, and the radius $SB$ of the axle; to find the radius of the wheel, so that $p$ may draw up $q$ through a given space in the least time possible, the inertia of the wheel and axle not being considered.

Let $SB=b$, $SA=x$; then the accelerating force upon $q$, by the preceding Problem, $=\frac{\overline{px-qb}\times b}{px^2+qb^2}$. Now $S \propto F\times T^2$;

therefore $T^2 \propto \frac{1}{F}$, when $S$ is given; hence, since the time is a minimum, the reciprocal of the force, or $\frac{px^2+qb^2}{\overline{px-qb}\times b}$, is a minimum; ∴ its fluxion $=0$;

$$\text{hence } 2px\dot{x}\times\overline{px-qb}-p\dot{x}\times\overline{px^2+qb^2}=0;$$
$$\therefore\ 2px^2-2qbx-px^2-qb^2=0,$$
$$\text{or } px^2-2qbx=qb^2,$$

from the solution of which quadratic $x=\frac{qb+\sqrt{q^2b^2+pqb^2}}{p}$.

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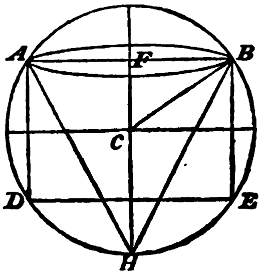
## SECTION II.

### PROB. 52.

To inscribe the greatest cylinder in a given sphere.

Let *ABED* be the cylinder; $CB=r$, $CF=x$, $BF=y$. Then $y^2=\overline{r^2-x^2}$; and the content $=2p\times\overline{r^2x-x^3}$, a maximum; $\therefore\ r^2\dot{x}=3x^2\dot{x}$, and $x=\frac{r}{\sqrt{3}}$.



### PROB. 53.

To inscribe the greatest cone in a given sphere.

Let *HAB* be the cone required. Then, since the content of a cone $=\frac{1}{3}$ of a cylinder of the same base and altitude, (Art. 51. Ex. 6.) the content of $HAB=p\times FB^2\times\frac{HF}{3}$. Let $CH=r$, $HF=x$; then $BF^2=2rx-x^2$; therefore $p\times$

$\frac{\overline{2rx-x^2}\times x}{3}$ is a maximum; hence $2rx^2-x^3$ is a maximum, and $4rx\dot{x}=3x^2\dot{x}$; $\therefore\ x=\frac{4r}{3}$.

## Prob. 54.

To inscribe the greatest cylinder in a given spheroid, the axis of the cylinder being supposed to coincide with the axis of the spheroid.

Let $a$ = the semi-major axis, $b$ the semi-minor, $y$ the ordinate, and $x$ that part of the axis intercepted between the ordinate and the center. Then $y^2=\frac{b^2}{a^2}\times\overline{a^2-x^2}$; $\therefore\ py^2\times 2x$, the content, $=\frac{2pb^2}{a^2}\times\overline{a^2x-x^3}$, a maximum; $\therefore\ a^2\dot{x}=3x^2\dot{x}$, and $x=\frac{a}{\sqrt{3}}$.

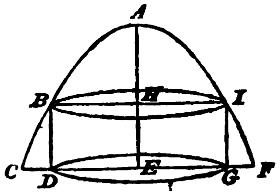
## Prob. 55.

To inscribe the greatest cone in a given spheroid, its vertex coinciding with the extremity of the major axis.

Let the part of the axis intercepted between the vertex of the cone and its base $=x$; then $y^2=\frac{b^2}{a^2}\times\overline{2ax-x^2}$; and the content of the cone $=\frac{pb^2}{3a^2}\times\overline{2ax^2-x^3}$, a maximum; therefore $4ax\dot{x}=3x^2\dot{x}$, and $x=\frac{4a}{3}$.

## Prob. 56.

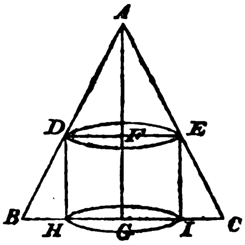
To inscribe the greatest cylinder in a given paraboloid *CAF*.



Let $BIGD$ be the cylinder; $AH = x$, $AE = b$, $HI = y$, and the latus rectum $= c$. Then $y^2 = cx$; $\therefore$ the content, or $py^2 \times HE, = pcx \times \overline{b - x}$, a maximum; hence $b\dot{x} = 2x\dot{x}$, and $x = \frac{b}{2}$.

### Prob. 57.

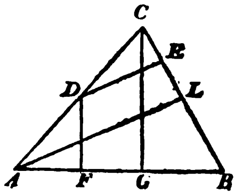
To inscribe the greatest cylinder in a given cone.



Let $AF = x$, $FE = y$, $AG = a$, $GB = b$; then $y = \frac{bx}{a}$; therefore the content of the cylinder, or $py^2 \times FG, = \frac{pb^2}{a^2} \times x^2 \times \overline{a - x}$, a maximum; hence $ax^2 - x^3$ is a maximum, and $2ax\dot{x} = 3x^2\dot{x}$, or $x = \frac{2a}{3}$.

### Prob. 58.

Find that point in the side of the triangle $ABC$, from which, if perpendiculars be drawn to the other two sides, their product may be a maximum.



Let $D$ be the point required; draw $DE$ perpendicular to $CB$, and $DF$ to $AB$; also from $A$ and $C$ draw $AL$ and $CG$ perpendicular to $CB$ and $AB$. Then by similar triangles, $ADF$, $ACG$, and $CDE$, $CAL$,

$$\begin{array}{c} AD : DF :: AC : CG \\ CD : DE :: CA : AL \\ \hline \end{array}$$

$$\therefore AD \times DC : DF \times DE :: AC^2 : CG \times AL.$$

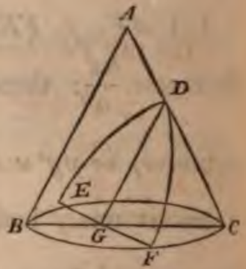
Now as the last two terms are constant in the same triangle, $DF \times DE \propto AD \times DC$, and is therefore a maximum when

$AD \times DC$ is a maximum; that is, when $AD = DC$ (Art. 23. Ex. 2.); hence $AC$ must be bisected in $D$, and $D$ is the point required.

This Figure answers for an acute-angled triangle; the process is nearly the same for a right and obtuse-angled triangle.

### Prob. 59.

From a given cone $ABC$, to cut the greatest parabola $DEF$.



Let $BC$ be that diameter of the base which is perpendicular to $EF$. Take $CG = x$, $CB = b$, $BA = a$; then, by the property of the circle, $EG = \sqrt{BG \times GC} = \sqrt{\overline{b-x} \times x} = \sqrt{bx - x^2}$; $\therefore\ EF = 2 \times \sqrt{bx - x^2}$. Also $CB : BA :: CG : GD$, or $b : a :: x : GD = \frac{ax}{b}$; $\therefore$ the area of the parabola $EDF = \frac{2}{3} \times \frac{ax}{b} \times 2 \times \sqrt{bx - x^2}$ (Art. 48. Ex. 2.), which varies as $x \times \sqrt{bx - x^2}$, a maximum; hence its square $bx^3 - x^4$ is a maximum, and its fluxion $= 0$; $\therefore\ 3bx^2\dot{x} = 4x^3\dot{x}$, and $x = \frac{3b}{4}$.

### Prob. 60.

The distance of the center of gravity from the vertex of a solid, formed by the revolution of a superficies of the parabolic kind, is $\frac{7}{8}$ of its axis; required the nature of the generating curve.

The fluent of $\frac{y^2x\dot{x}}{f.\,y^2\dot{x}} = \frac{7}{8}x$. (Art. 61.)

Assume an equation $a^{n-1}x = y^n$; then $y^2 = a^{\frac{2n-2}{n}}x^{\frac{2}{n}}$, and

$$\frac{f.\,y^2x\dot{x}}{f.\,y^2\dot{x}} = \frac{f.\,a^{\frac{2n-2}{n}}x^{\frac{2}{n}+1}\dot{x}}{f.\,a^{\frac{2n-2}{n}}x^{\frac{2}{n}}\dot{x}} = \frac{\overline{\frac{n+2}{n}} \times x^{\frac{2n+2}{n}}}{\frac{2n+2}{n} \times x^{\frac{n+2}{n}}} = \frac{n+2}{2n+2} \times x, \text{ or}$$

$\frac{n+2}{2n+2}=\frac{7}{8}$; $\therefore 8n+16=14n+14$, and $6n=2$, and $n=\frac{1}{3}$; $\therefore$ the equation is $x=a^{\frac{2}{3}}y^{\frac{1}{3}}$, or $x^3=a^2y$.

In the common parabola, $\frac{n+2}{2n+2}=\frac{2}{3}$; $\therefore 3n+6=4n+4$, and $n=2$; $\therefore ax=y^2$, the equation.

### Prob. 61.

Required to find the nature of the curve, in which the sub-tangent : the sub-normal :: $m^2x^2 : y^2$.

By the Problem, $\frac{y\dot{x}}{\dot{y}} : \frac{y\dot{y}}{\dot{x}} :: m^2x^2 : y^2$;

$\therefore \dot{x}^2 : \dot{y}^2 :: m^2x^2 : y^2$, and $\dot{x} : \dot{y} :: mx : y$;

$\therefore \frac{\dot{x}}{x}=\frac{m\dot{y}}{y}$, and hyp. log. $x=m\times$ hyp. log. $y$; $\therefore x \propto y^m$, and $a^{m-1}x=y^m$, the equation.

### Prob. 62.

Required the equation to a curve, whose sub-tangent $=n$ times its abscissa.

Here $\frac{y\dot{x}}{\dot{y}}=nx$; $\therefore \frac{\dot{x}}{x}=n\times\frac{\dot{y}}{y}$, and hyp. log. $x=n\times$ hyp. log. $y$; $\therefore x\propto y^n$, or $a^{n-1}x=y^n$.

If $n=2$, $ax=y^2$, the equation to the parabola.

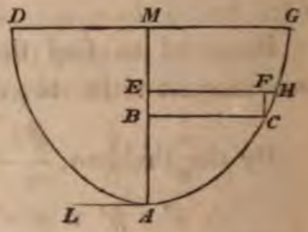
### Prob. 63.

To find the nature of the curve, whose tangent is a given quantity.

Let $x=$ the abscissa, and $y$ the ordinate; let the tangent $=b$; then, since the sub-tangent $=\frac{y\dot{x}}{\dot{y}}$, we have $\frac{y^2\dot{x}^2}{\dot{y}^2}+y^2=b^2$; $\therefore y^2\times\overline{\dot{x}^2+\dot{y}^2}=b^2\dot{y}^2$, or $y^2\dot{z}^2=b^2\dot{y}^2$; $\therefore \dot{z}=\frac{b\dot{y}}{y}$, and $z\propto b\times$ hyp. log. $y$; or $az=b\times$ hyp. log. $y$, the equation.

### Prob. 64.

To determine the law of the weights, which press upon each particle of a perfectly flexible line, so that it shall form a curve whose equation is $a^3x=y^4$.



Let $DAG$ represent the curve, $AM$ its axis, and $A$ the lowest point. Draw $EH$, $BC$, two ordinates to the axis, indefinitely near each other; let $CF$ be drawn perpendicular to $EH$, and $AL$ be a tangent at $A$. Then $AC$ being considered as inflexible, after it has assumed the proper situation, it is kept at rest by three forces; at $A$ by the action of $AD$ in the direction $AL$; at $C$ by the part of the line $CG$ in a tangential direction, and by the pressure in the direction $FC$; hence, if the effect of $AD=b$, $AB=x$, $BC=y$,

$$\dot{y} : \dot{x} :: b : \text{pressure}; \therefore \dot{x}=\frac{Pr\times\dot{y}}{b};$$

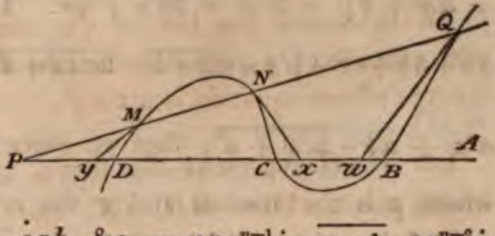
but by the nature of the curve, since $a^3x=y^4$, $\dot{x}=\dfrac{4y^3\dot{y}}{a^3}$;

$$\therefore \frac{Pr\times\dot{y}}{b}=\frac{4y^3\dot{y}}{a^3}, \text{ and the pressure} \propto y^3.$$

### Prob. 65.

Let $AP$ be the abscissa of any curve, $PMNQ$ an ordinate revolving about the fixed point $P$, and cutting the curve in as many points as it has dimensions; $My$, $Nx$, and $Qw$, being drawn tangents to the curve in $M$, $N$, and $Q$; it is required to find the sum of the reciprocal sub-tangents $\dfrac{1}{Py}+\dfrac{1}{Px}+\dfrac{1}{Pw}$.

Let the equation to the curve be $y^n-\overline{a'x+b'}\times y^{n-1}+\overline{c'x^2+d'x+e'}\times y^{n-2}-\text{\&c.}+px^n-qx^{n-1}+rx^{n-2}-\text{\&c.}=0$;



let $a$, $b$, $c$ be the values of $y$ corresponding to $AP$ or $x$, the abscissa; then, since the last term is the product of all the roots with their signs changed, $a \times b \times c \times$ &c. $= px^n - qx^{n-1} +$ &c.; hence $\dot{a}bc$, &c. $+ \dot{b}ac$, &c. $+ \dot{c}ab$, &c. $= npx^{n-1}\dot{x} - \overline{n-1} \,.\, qx^{n-2}\dot{x}$ + &c.; $\therefore$ dividing the former part of the equation by $a \times b \times c \times$ &c., and the latter by $px^n - qx^{n-1} +$ &c., we have,

$$\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} + \&\text{c.} = \frac{npx^{n-1}\dot{x} - \overline{n-1} \,.\, qx^{n-2}\dot{x} + \&\text{c.}}{px^n - qx^{n-1} + \&\text{c.}};$$

hence, $$\frac{\dot{a}}{a\dot{x}} + \frac{\dot{b}}{b\dot{x}} + \frac{\dot{c}}{c\dot{x}} + \&\text{c.} = \frac{npx^{n-1} - \overline{n-1} \,.\, qx^{n-2} + \&\text{c.}}{px^n - qx^{n-1} + \&\text{c.}};$$

that is, $$\frac{1}{Py} + \frac{1}{Px} + \frac{1}{Pw} + \&\text{c.} = \frac{npx^{n-1} - \overline{n-1} \,.\, qx^{n-2} + \&\text{c.}}{px^n - qx^{n-1} + \&\text{c.}} =$$

$$\frac{nx^{n-1} - \overline{n-1} \,.\, \frac{q}{p} x^{n-2} + \&\text{c.}}{x^n - \frac{qx^{n-1}}{p} + \&\text{c.}}.$$

Cor. Since the roots of the equation $x^n - \frac{q}{p} x^{n-1} +$ &c. $= 0$, are $AB$, $AC$, $AD$, &c., the coefficients of this equation are constant; also $x$ is constant, because $P$ is a fixed point by the hypothesis; $\therefore$ $$\frac{nx^{n-1} - \overline{n-1} \,.\, \frac{q}{p} x^{n-2} + \&\text{c.}}{x^n - \frac{q}{p} x^{n-1} + \&\text{c.}}$$ is constant; that is, the sum of the reciprocal sub-tangents is a constant quantity.

## Prob. 66.

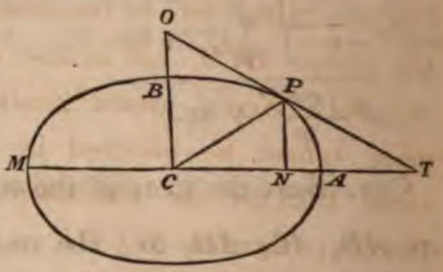
**In the same curve, to find the sum of the sub-normals.**

If $p$ and $q$ be the coefficients of the second and third terms of an equation, the sum of the squares of the roots $= p^2 - 2q$;

hence, on the supposition in the last Problem, $a^2+b^2+c^2+$ &c. $=\overline{a'x+b'}|^2-\overline{2c'x^2+2d'x+2e'}$. Take the fluxion; then $a\dot{a}+b\dot{b}+c\dot{c}=a'\dot{x}\times\overline{a'x+b'}-\overline{2c'x\dot{x}+d'\dot{x}}$; $\therefore\ \frac{a\dot{a}}{\dot{x}}+\frac{b\dot{b}}{\dot{x}}+\frac{c\dot{c}}{\dot{x}}=$ $a'^2x+a'b-\overline{2c'x+d'}$; that is, since the sub-normal $=\frac{y\dot{y}}{\dot{x}}$, where $x$ is the abscissa and $y$ the ordinate, in this case the sum of the sub-normals $=a'^2x+a'b-\overline{2c'x+d'}$.

## Prob. 67.

To draw a tangent to an ellipse, so that the triangle contained under this tangent, and the major and minor semi-axes produced, may be a minimum.



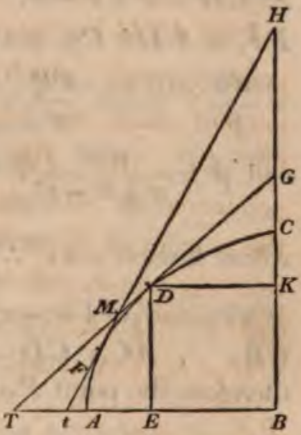
Let $ABM$ be the ellipse, $C$ the center. Let $P$ be the point through which the tangent must be drawn; join $CP$; draw $PN$ perpendicular to $AC$, and let the tangent at $P$ meet the two axes produced in $O$ and $T$. Take $AC=a$, $CB=b$, $CN=x$; then $PN=\frac{b}{a}\times\sqrt{a^2-x^2}$, and $CT=\frac{a^2}{x}$ by conic sections; $\therefore NT$ $=\frac{a^2}{x}-x=\frac{a^2-x^2}{x}$; and by similar triangles, $TNP$, $TCO$, $TN:NP::TC:CO$, or $\frac{a^2-x^2}{x}:\frac{b}{a}\times\sqrt{a^2-x^2}::\frac{a^2}{x}:$ $CO=\frac{ab}{\sqrt{a^2-x^2}}$; therefore $CO\times CT$, or $\frac{a^3b}{x\times\sqrt{a^2-x^2}}$, is a minimum; hence $x\times\sqrt{a^2-x^2}$, or $a^2x^2-x^4$, is a maximum; $\therefore\ 2a^2x\dot{x}-4x^3\dot{x}=0$, and $x=\frac{a}{\sqrt{2}}$.

Cor. Since $x=\frac{a}{\sqrt{2}}$, $CO\left(=\frac{ab}{\sqrt{a^2-x^2}}\right)=b\times\sqrt{2}$; and $NP\left(=\frac{b}{a}\times\sqrt{a^2-x^2}\right)=\frac{b}{\sqrt{2}}$; $\therefore NP : CO :: \frac{b}{\sqrt{2}} : b\times\sqrt{2} :: 1 : 2$; $\therefore TP : TO :: 1 : 2$, or the tangent $TO$ is bisected in $P$. The same is true for any oval figure.

## Prob. 68.

The greatest parallelogram that can be inscribed in a curve $ABC$ concave to its axis, and the least triangle that can be described about it will be when the sub-tangent $ET$ is equal to the base $BE$ of the parallelogram, or half the base of the triangle.

By Art. 23, the greatest parallelogram which can be inscribed in the triangle $BTG$, has its base $BE = \frac{1}{2} BT$. Now a greater parallelogram cannot be inscribed in the curve than in the circumscribing triangle; therefore $BEDK$ is the greatest parallelogram which can be inscribed in the curve.

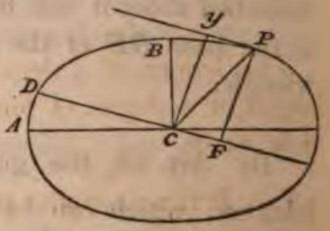


Also if $TG$ be bisected in $D$, where it touches the curve, $BTG$ is the least triangle; if not, let $BtH$ be less, and suppose $tH$ to cut $TG$ in $M$, and to touch the curve in $F$. Then, since $DG=DT$, $MG$ is greater than $MT$; and $Mt$ being less than $MT$, and consequently less than $MG$, must be much less than $MH$. Hence, since the vertical angles at $M$ are equal, the triangle $HMG$, which we have added to the original triangle, is greater than $TMt$, which we have taken away; that is, the triangle $BHt$ is greater than $BGT$.

The same kind of proof may be extended to any rectilinear figure, which circumscribes an oval; for the other sides being supposed constant, the figure is always the least when the remaining side is bisected in the point of contact.

### Prob. 69.

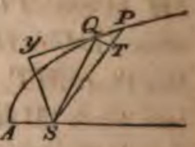
To find that point $P$ in an ellipse, to which if a tangent be drawn, the part $Py$, intercepted between the point $P$ and the perpendicular $Cy$ drawn from the center upon the tangent, may be a maximum.



Draw the conjugate diameter $CD$, and draw $PF$ perpendicular to it. Let $Cy=p$, $PC=x$, $AC=a$, $BC$, the semi-minor, $=b$. Then $CD^2+(CP^2)\,x^2=a^2+b^2$; $\therefore CD^2=a^2+b^2-x^2$. Now $CD\times PF$, or $CD\times Cy$, $=a\times b$; $\therefore CD^2=\frac{a^2b^2}{p^2}$; hence $\frac{a^2b^2}{p^2}=a^2+b^2-x^2$, and $p^2=\frac{a^2b^2}{a^2+b^2-x^2}$; $\therefore Py^2=CP^2-Cy^2=x^2-\frac{a^2b^2}{a^2+b^2-x^2}$, whose fluxion $=0$; $\therefore 2x\dot{x}-\frac{a^2b^2\times 2x\dot{x}}{\overline{a^2+b^2-x^2}|^2}=0$; $\therefore \overline{a^2+b^2-x^2}|^2=a^2b^2$, and $a^2+b^2-x^2=ab$; that is, $CD^2+x^2-x^2=AC\times CB$; $\therefore AC:CD::CD:CB$; hence $x$ is known, and therefore the point $P$.

### Prob. 70.

To find the area of the parabola, considered as a spiral.



Let $PQ$ represent the fluxion of the curve, $Sy$ a perpendicular on the tangent at $P$; join $SP$, $SQ$, and draw $QT$ perpendicular to $SP$. Then, if $SA=a$, and $SP=x$, $Sy=\sqrt{ax}$, $PT=\dot{x}$, and $SQP$ is the fluxion of the area $SAQ$. Now $SPQ=\frac{SP\times QT}{2}$; and $QT:TP$

or $\dot{x}$ :: $Sy$ or $\sqrt{ax}$ : $Py$ or $\sqrt{x^2-ax}$; $\therefore QT = \frac{a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}{\sqrt{x^2-ax}}$ $= \frac{a^{\frac{1}{2}}\dot{x}}{\sqrt{x-a}}$; and the fluxion of the area $= \frac{a^{\frac{1}{2}}x\dot{x}}{2\sqrt{x-a}}$, and $f. = \frac{a^{\frac{1}{2}}}{2} \times : \frac{2x \times \sqrt{x-a}}{3} + \frac{4a \times \sqrt{x-a}}{3}$ (Fluent 19.) $=$ the area $SAP$; the corr. $= 0$.

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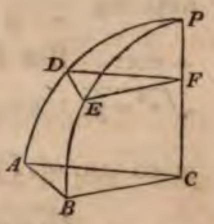
## ON MERCATOR's PROJECTION OF THE SPHERE.

### LEMMA.

The length of a degree of latitude at any place is to the length of a degree of longitude there as radius to the cosine of latitude.

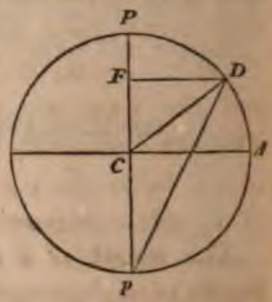
Let $P$ represent the pole, and $C$ the center of the earth; $AH$ the equator, $AB$ the length of a degree of longitude at the equator; $PDA$, $PEB$, two meridians; from $F$ draw in the planes $PAC$, $PBC$, the lines $FD$, $FE$ parallel to $AC$, $BC$. The included arc $DE$ is part of a small circle parallel to $AB$, and measures a degree in longitude at $D$. Now by similar sectors $ACB$, $DFE$; $AB : DE :: AC : DF$; or, since $AB =$ the length of a degree of latitude at any place $D$, the length of a degree of latitude : the length of a degree of longitude :: radius : cosine of latitude.



In Mercator's Projection, the sphere is projected upon a plane, $P$ is at an infinite distance, and the meridians $PA$ and $PB$ are parallel. Hence, in all latitudes $DE$ is the same; therefore to preserve the just ratio between a degree of latitude and longitude in the projection, the degrees of latitude must increase in receding from the equator, according to the proportion in the Lemma.

### Prob. 71.

In this projection, to find the length of an arc of the meridian corresponding to any given latitude.

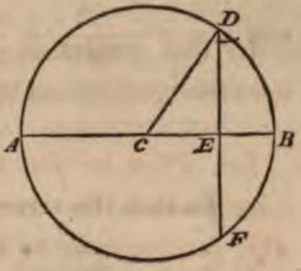


Let $PCp$ be the axis of the earth, $AC$ the equator, $D$ some point in the meridian $PA$; draw $DF$ perpendicular to $Pp$, and join $Dp$, $DC$. Then, if $AC=r$, $AD=x$, $CF=y$, and the length of the projection of $AD$ on the plane $=z$, we have by the Lemma, $\dot{z} : \dot{x} :: r : \sqrt{r^2-y^2}$; $\therefore \dot{z}=\frac{r\dot{x}}{\sqrt{r^2-y^2}}$. But (by Art. 44.) $\dot{x}=\frac{r\dot{y}}{\sqrt{r^2-y^2}}$; $\therefore \dot{z}=\frac{r^2\dot{y}}{r^2-y^2}=\frac{r}{2}\times\frac{2r\dot{y}}{\sqrt{r^2-y^2}}$, and $z=\frac{r}{2}\times$ hyp. log. $\frac{r+y}{r-y}$ (Art. 43.) + corr. $= r\times$ hyp. log. $\sqrt{\frac{r+y}{r-y}}$ + corr. But $DF : Fp :: r :$ tan. of the angle $FDp$, that is, $\sqrt{r^2-y^2} : r+y :: r : T, FDp$; $\therefore$ the tangent of $FDp$, or co-tan. of $FpD$, or of $\frac{1}{2}$ $DCP$, the co-lat. $=r\times\sqrt{\frac{r+y}{r-y}}$; hence $\sqrt{\frac{r+y}{r-y}}=\frac{\text{co-tan. of } \frac{1}{2} \text{ the co-lat.}}{r}$, and $z=r\times$ hyp. log. $\frac{\text{co-tan. of } \frac{1}{2} \text{ co-lat.}}{r}+C$. Now when $z=0$, the co-tan. of $\frac{1}{2}$ the co-lat. is the co-tan. of 45°, and equals $r$; hence $C=-r\times$ hyp. log. $\frac{r}{r}=0$; therefore $z=r\times$ hyp. log. $\frac{\text{co-tan. of } \frac{1}{2} \text{ co-lat.}}{r}$.

### Prob. 72.

Given the arc of a circle, to find its sine and cosine.

Let $BD$ the given arc $=z$, $DE$ its sine $=y$, $CE$ its cosine $=x$. Then, by the nature of the circle, $\dot{z} : -\dot{x} :: r : y$; therefore $y=\frac{-r\dot{x}}{\dot{z}}$. Now to the cosine $CE$ we have two values of $z$, $BD$ and $BF$, which are equal to each other, but they must have different signs; if $z=a$, $z-a=0$; if $z=-a$, $z+a=0$, and the quadratic resulting from these two is $z^2-a^2=0$.



The same reasoning holds for any other corresponding values of $z$; hence the equation, whose roots are the several values of $z$, will contain only its even powers; also, if $CE=CB=r$, $z=0$; therefore if $CE$ be assumed in a series in terms of $z$, the first term will be $r$, and the succeeding terms will contain the even powers of $z$.

$$\text{Let } x=r+az^2+bz^4+cz^6+\text{ \&c.}$$
$$\text{then } \dot{x}=2az\dot{z}+4bz^3\dot{z}+6cz^5\dot{z}+\text{ \&c.}$$
$$\therefore y\left(=-\frac{r\dot{x}}{\dot{z}}\right)=-2arz-4brz^3-6crz^5-\text{ \&c.}$$
$$\text{and } \dot{y}=-2ar\dot{z}-3.4brz^2\dot{z}-5.6crz^4\dot{z}-\text{ \&c.}$$

But, in a circle, $\dot{z} : \dot{y} :: r : x'$; $\therefore x\dot{z}-r\dot{y}=0$;

$$\text{hence } \left.\begin{matrix} r\dot{z}+az^2\dot{z}+bz^4\dot{z}+\text{ \&c.} \\ +2ar^2\dot{z}+3.4.br^2\dot{z}+5.6.cr^2z^4\dot{z}+\text{ \&c.}\end{matrix}\right\}=0.$$

Therefore, by equating the coefficients of the corresponding terms,

$$r+2ar^2=0;\ \therefore a=-\frac{1}{2r}.$$
$$a+3.4br^2=0;\ \therefore b=-\frac{a}{3.4.r^2}=\frac{1}{2.3.4r^3}.$$
$$b+5.6cr^2=0;\ \therefore c=-\frac{b}{5.6.r^2}=-\frac{1}{2.3.4.5.6r^5},\ \text{\&c.};$$

whence, by substituting for $a$, $b$, &c., their values in the assumed equation, $x=r-\frac{z^2}{2r}+\frac{z^4}{2.3.4.r^3}-\frac{z^6}{2.3.4.5.6r^5}+$ &c., and $y=z-\frac{z^3}{2.3r^2}+\frac{z^5}{2.3.4.5r^4}-$ &c.

## PROB. 73.

To sum the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+$ &c. ad inf.

In the last Problem, if $r=1$, $y=z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-$ &c.; let $y=0$; then $z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-$ &c. $=0$, in which one root $z=0$; hence, dividing by $z$, $1-\frac{z^2}{2.3}+\frac{z^4}{2.3.4.5}-$ &c. $=0$. Since $y=0$, if $C=$ the semi-circumference, the other values of $z$ are $1C$, $2C$, $3C$, &c. $-1C$, $-2C$, $-3C$, &c., each series ad inf. Let $v=\frac{1}{z}$; then $1-\frac{1}{2.3.v^2}+\frac{1}{2.3.4.5.v^4}-$ &c. $=0$; or, multiplying by $v^n$,

$$v^n-\frac{v^{n-2}}{2.3}+\frac{v^{n-4}}{2.3.4.5}-\text{\&c.}=0,$$

in which $n$ values of $y=0$; and the other values are $\frac{1}{1C}$, $\frac{1}{2C}$, $\frac{1}{3C}$, &c. ad inf., and $-\frac{1}{1C}$, $-\frac{1}{2C}$, $-\frac{1}{3C}$, &c. ad inf. Now the sum of the squares of the roots of this equation $=p^2-2q$; here, as the second term is wanting, $p=0$, and $q=-\frac{1}{2.3}$; $\therefore p^2-2q=\frac{1}{3}$; that is, since the squares of $\frac{1}{1C}$, $\frac{1}{2C}$, &c. are the same as the squares of $-\frac{1}{1C}$, $-\frac{1}{2C}$, &c., $\frac{2}{1^2C^2}+\frac{2}{2^2C^2}+\frac{2}{3^2C^2}+$ &c. ad inf. $=\frac{1}{3}$; $\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+$ &c. ad inf. $=\frac{C^2}{6}$.

Cor. 1. In the same manner the sums of any of the even powers of the reciprocals of the natural numbers may be found. Instead of finding the value of $p^2-2q$, take the algebraic expression required in that particular case. The sum of the reciprocals of the odd powers cannot be determined by this method, as the odd powers of the negative roots destroy the odd powers of the positive.

Cor. 2. Since $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.=\frac{C^2}{6}$,

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\&c.+\frac{1}{2^2}+\frac{1}{4^2}+\&c.=\frac{C^2}{6};$$

that is, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\&c.+\frac{1}{2^2}\times\overline{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\&c.}=\frac{C^2}{6}$,

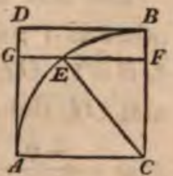
$$\text{or } \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\&c.+\frac{1}{2^2}\times\frac{C^2}{6}=\frac{C^2}{6};$$

$$\therefore\ \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\&c.=\frac{C^2}{6}-\frac{C^2}{24}=\frac{C^2}{8}.$$

In the same manner the sum of the reciprocals of all the even powers of 1, 3, 5, 7, &c. may be found.

### Prob. 74.

To compare the momenta of a sphere and its circumscribed cylinder, whilst they revolve round a common axis.



Let *AEBC* be the quadrant of a circle, whose semi-diameter is *CA* or *CB*. Complete the square *ACBD*; draw any line *GEF* parallel to *AC*, cutting *AD*, *BC*, in *G* and *F*, and the quadrantile arc in *E*. Join *EC*. Now let the segments *AEFC* of the quadrant, and *AGFC* of the square, revolve about the axis *CF*, and generate segments of the sphere and circumscribed cylinder. Take $CF=x$, $FE=y$, $CA=r$. Then, the momenta being as the quantities of matter and the velocity jointly, the momentum of the spherical segment $AEFC$ : the momentum of the cylindrical segment $AGFC$ :: $f.\, y^2\dot{x}$ : $f.\, r^3\dot{x}$.

$$:: f.\, \dot{x} \times \overline{r^2 - x^2}\big|^{\frac{3}{2}} : f.\, r^3\dot{x}$$

$$:: f.\, r^2\dot{x} \times \sqrt{r^2 - x^2} - f.\, x^2\dot{x} \times \sqrt{r^2 - x^2} : f.\, r^3\dot{x}.$$

Now the $f.\, r^2\dot{x} \times \sqrt{r^2 - x^2} = r^2 \times$ area $ACFE$; and $f.\, x^2\dot{x} \times \sqrt{r^4 - x^2} = \dfrac{r^2 \times \text{area } ACFE - x \times \overline{r^2 - x^2}\big|^{\frac{3}{2}}}{4}$ (Fluent 24); $\therefore$ the third term in this proportion $= \dfrac{3}{4} r^2 \times$ area $CFEA$ + $\dfrac{x \times \overline{r^2 - x^2}\big|^{\frac{3}{2}}}{4}$. Let $x = r$; and we have, the momentum of the sphere : the momentum of the cylinder :: $\dfrac{3r^2}{4} \times$ quadrant $AEBC$ : $r^4$; or (if $Q =$ the arc of the quadrant)

$$:: \frac{3r^3}{8} \times Q : r^4$$

$$:: 3Q : 8r.$$

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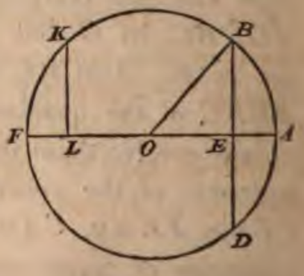
## SECTION III.

### Prob. 75.

To resolve $v^{2n} - 2xv^n + 1 = 0$ into its quadratic divisors, $x$ being equal to or less than unity.



Let $AB$ and $AK$ be two circular arcs, which are to each other as $1 : n$; let $AB = z$, $AK = nz$, $OB = 1$, $OE$ the cosine of $AB = y$, and $OL$ the cosine of $AK = x$. Then $\dot{z} = \dfrac{-\dot{y}}{\sqrt{1 - y^2}}$ (Art. 44), and $n\dot{z} = \dfrac{-\dot{x}}{\sqrt{1 - x^2}}$; $\therefore \dfrac{\dot{x}}{\sqrt{1 - x^2}} = \dfrac{n\dot{y}}{\sqrt{1 - y^2}}$; and multiplying each denominator by

$\sqrt{-1}$, $\frac{\dot{x}}{\sqrt{x^2-1}} = \frac{n\dot{y}}{\sqrt{y^2-1}}$; $\therefore$ hyp. log. $x + \sqrt{x^2-1} =$ $n \times$ hyp. log. $y+\sqrt{y^2-1}$ (Art. 43); hence $x + \sqrt{x^2-1} = \overline{y+\sqrt{y^2-1}}|^n$, by the nature of logarithms.

Now let $v=y+\sqrt{y^2-1}$; $\therefore$ $v-y=\sqrt{y^2-1}$, and $v^2-2yv + y^2=y^2-1$; $\therefore$ $v^2-2yv+1=0$. Also $v^n=\overline{y+\sqrt{y^2-1}}|^n$ $=x+\sqrt{x^2-1}$; $\therefore$ $v^n-x=\sqrt{x^2-1}$, and $v^{2n}-2xv^n+1=0$, the given equation; and since $v$ is the same in both equations, one quadratic divisor is $v^2-2yv+1=0$.

Now $x$ is not only the cosine of $AK$, but of $360+AK$, of $2\times 360+AK$, &c.; $\therefore$ $y$ is not only the cosine of $\frac{AK}{n}$, but of $\frac{360+AK}{n}$, of $\frac{2\times 360+AK}{n}$, &c. Call these cosines $a$, $b$, $c$, &c.; then $v^{2n}-2xv^n+1=\overline{v^2-2av+1}\times\overline{v^2-2bv+1}\times$ $\overline{v^2-2cv+1}\times$ &c.

There can only be $n$ different values of $y$; for after taking $n$ arcs, $\frac{AK}{n}$, $\frac{360+AK}{n}$, &c., the same cosines will recur.

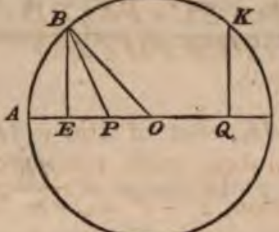
Cor. 1. If $AK$ be taken equal to the whole or to half of the circumference, the equation is $v^{2n}\mp 2v^n+1=0$, and its square root is $v^n\mp 1=0$. But every equation, which is a square, has another root equal to each of its roots; therefore the roots of $v^n\mp 1=0$ are found in the same manner.

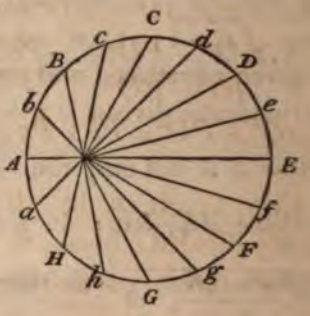
Cor. 2 The quadratic divisors of $v^{2n}-2xr^nv^n+r^{2n}=0$ are found in the same way; for this equation is merely the equation $v^{2n}-2xv^n+1=0$, having its roots multiplied by $r$. Hence, multiply the roots of the above quadratic divisors by $r$, and we have $v^2-2arv+r^2=0$, $v^2-2brv+r^2=0$, &c. for the required divisors.

### Prob. 76.

### To demonstrate Cotes' Properties of the Circle.

1. If any point $P$ be taken in the radius of a circle $AO$ or the radius produced, and the circumference be divided into $n$ equal parts in $B$, $C$, $D$, &c.; then $AO^n - PO^n$, or $PO^n - AO^n = \times PB$, $PC \times PD \times$ &c.





By Prob. 75, the quadratic divisors of $v^{2n} - 2xv^n + 1 = \overline{v^2 - 2av + 1}$, $\overline{v^2 - 2bv + 1}$, &c. Assume a point $P$ in the radius, and draw $PB$ to the circumference; let $v = PO$, $y = OE$, and the radius $= 1$. Then $BO^2 = OP^2 + PB^2 + 2OP \times PE$, or $1^2 = v^2 + BP^2 + 2v \times \overline{y - v}$; $\therefore BP^2 = 1^2 - v^2 - 2v \times \overline{y - v} = v^2 - 2yv + 1$. Also $y$ is the cosine of $\frac{AK}{n}$, $\frac{AK + 360}{n}$, $\frac{AK + 2 \times 360}{n}$, &c., whose cosines are $a$, $b$, $c$, &c., and $\overline{v^2 - 2av + 1} \times \overline{v^2 - 2bv + 1} \times$ &c. $= v^{2n} - 2xv^n + 1$. Let $AK =$ the whole circumference $C$; then these arcs are $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c., or $\frac{1^{\text{th}}}{n}$, $\frac{2^{\text{th}}}{n}$, $\frac{3^{\text{th}}}{n}$, &c. parts of the circumference; that is, if the circumference be divided into $n$ equal parts in $B$, $C$, $D$, &c., the cosines of $AB$, $AC$, $AD$, &c. are $a$, $b$, $c$, &c. and $x = 1$. Hence $PB^2 = \overline{v^2 - 2av + 1}$, $PC^2 = \overline{v^2 - 2bv} + 1$, &c.; $\therefore PB^2 \times PC^2 \times PD^2 \times$ &c. $= v^{2n} - 2v^n + 1$, and $PB \times PC \times PD \times$ &c. $= v^n - 1^n$, or $1^n - v^n = PO^n - AO^n$, or $AO^n - PO^n$, according as $P$ is without or within the circle.

### Second Property.

If the whole circumference be divided into $2n$ equal parts in $b$, $c$, $d$, &c., then $AO^n + PO^n = Pb \times Pc \times Pd \times$ &c.

By the preceding case, if $P$ be taken, for example, within the circle, $AO^{2n} - PO^{2n} = Pb \times PB \times Pc \times PC \times Pd \times PD$ $\times$ &c.; but $AO^n - PO^n = PB \times PC \times PD \times$ &c.;

$$\therefore \frac{AO^{2n} - PO^{2n}}{AO^n - PO^n} = \frac{Pb \times PB \times Pc \times PC \times Pd \times PD \times \text{\&c.}}{PB \times PC \times PD \times \text{\&c.}},$$

or $AO^n + PO^n = Pb \times Pc \times Pd \times$ &c.

Cor. 1. If $n$ be an even number, each semi-circle is equally divided; hence, in the equation, $1^n - v^n = \sqrt{v^2 - 2av + 1} \times \sqrt{v^2 - 2bv + 1} \times$ &c. one value of $y$ is $+1$, and another $-1$; therefore $1^n - v^n = \overline{1 - v} \times \sqrt{v^2 - 2av + 1} \times \sqrt{v^2 - 2bv + 1} \times$ &c. $\overline{1 + v}$. But each $PB$ has a $PH$ in the other semi-circle corresponding to it; therefore $1^n - v^n = \overline{1 - v^2} \times \overline{1 - 2av + v^2} \times \overline{1 - 2bv + v^2} \times$ &c. to $\frac{n}{2}$ terms. This is the method of finding the $n$ roots of 1; two roots are in this case $+1$ and $-1$, obtained by solving the quadratic $1 - v^2$, and all the others obtained from the remaining quadratics are impossible.

Cor. 2. If $n$ be an odd number, none of the points of section will fall on the extremity of the semi-circle, and therefore no cosine can equal $-1$; but the cosine of $\frac{n \times C}{n}$, or of 360°, $= 1$; $\therefore$ here $1^n - v^n = \overline{1 - v} \times \overline{1 - 2av + v^2} \times \overline{1 - 2bv + v^2} \times$ &c. $\frac{n+1}{2}$ terms. In this case, one value of $v$ in the equation $1^n - v^n = 0$, is 1; the other roots are impossible.

Cor. 3. By the second property, $1^n + v^n = \sqrt{1 - 2av + v^2} \times \sqrt{1 - 2bv + v^2} \times$ &c. to $n$ terms. Let $n$ be even; here, as in the former case, to every *Pb* in one semi-circle there is a correspondent *Pa* in the other; but no point of section falls on the extremity of the semi-circle; therefore $1^n+v^n=\overline{1-2av+v^2}\times\overline{1-2bv+v^2}\times$ &c. to $\frac{n}{2}$ terms. By solving these quadratic divisors of $1^n+v^n=0$, we obtain the $n$ roots of $-1$, and they are all impossible when $n$ is even.

Cor. 4. Let $n$ be odd. Since the arcs on each side of $A$ are equal, and no point of section fell upon the extremity of the semi-circle in the last Corollary, one arc will now be a semi-circle, and one value of $v=-1$; hence $1^n+v^n=\overline{1+v}\times\overline{1-2av+v^2}\times\overline{1-2bv+v^2}\times$ &c. $\frac{n+1}{2}$ terms. In this case, therefore, one root of the equation $1^n+v^n=0$ is $-1$, and the rest are impossible.

## Prob. 77.

To resolve $\frac{1}{v^{2n}-2xv^n+1}$, $x$ being less than 1, into its quadratic divisors.

Let $\frac{1}{v^{2n}-2xv^n+1}=\frac{A}{m-v}+\frac{B}{p-v}+\frac{C}{q-v}+$ &c.; then $A\times\overline{p-v}\times\overline{q-v}\times$ &c. $+B\times\overline{m-v}\times\overline{q-v}\times$ &c. $+C\times\overline{m-v}\times\overline{p-v}\times$ &c. $=1$. Substitute $m$ for $v$; then $A=\frac{1}{\overline{p-m}\times\overline{q-m}\times \text{&c.}}$. Substitute $p$ for $v$; then $B=\frac{1}{\overline{m-p}\times\overline{q-p}\times \text{&c.}}$. Now as $v^{2n}-2xv^n+1=\overline{m-v}\times\overline{p-v}\times$ &c., take the fluxion on each side, and divide by $\dot{v}$; then $-2nv^{2n-1}+2nxv^{n-1}=\overline{m-v}\times\overline{p-v}\times$ &c. $+\overline{p-v}\times\overline{q-v}\times$ &c. Hence, if $m$ be written for $v$, $-2nm^{2n-1}+2nxm^{n-1}=\overline{p-m}\times\overline{q-m}\times$ &c.; but $A=\frac{1}{\overline{p-m}\times\overline{q-m}\times \text{&c.}}$; therefore $A=$

$\frac{1}{2nxm^{n-1} - 2nm^{2n-1}} = \frac{m}{2nxm^{n} - 2nm^{2n}}$; and $B = \frac{p}{2nxp^{n} - 2np^{2n}}$, or $A = \frac{1}{2n} \times \frac{m}{xm^{n} - m^{2n}}$, and $B = \frac{1}{2n} \times \frac{p}{xp^{n} - p^{2n}}$. But by supposition, $m^{2n} - 2xm^{n} + 1 = 0$; $\therefore$ $xm^{n} - m^{2n} = 1 - xm^{n}$. Hence $A = \frac{1}{2n} \times \frac{m}{1 - xm^{n}}$, and $B = \frac{1}{2n} \times \frac{p}{1 - xp^{n}}$. Now $\frac{A}{m - v} + \frac{B}{p - v} = \frac{pA + mB - \overline{A + B} \times v}{mp - \overline{m + p} \times v + v^2}$; and $1 - 2av + v^2$, a quadratic divisor of $1 - 2xv^{n} + v^{2n}$, is equal to $\overline{m - v} \times \overline{p - v}$; therefore $mp = 1$, and $m + p = 2a$; and we have $\frac{A}{m - v} + \frac{B}{p - v} = \frac{pA + mB - \overline{A + B} \times v}{1 - 2av + v^2}$. The object then is to calculate $pA + mB$ and $A + B$. Now $pA = \frac{1}{2n} \times \frac{pm}{1 - xm^{n}}$, or $= \frac{1}{2n} \times \frac{1}{1 - xm^{n}}$; and $mB = \frac{1}{2n} \times \frac{1}{1 - xp^{n}}$; therefore $pA + mB = \frac{1}{2n} \times : \frac{1}{1 - xm^{n}} + \frac{1}{1 - xp^{n}} = \frac{1}{2n} \times \frac{2 - x \times \overline{m^{n} + p^{n}}}{1 - x . \overline{m^{n} + p^{n}} + x^2 m^{n} p^{n}}$. But $m^{2n} - 2xm^{n} + 1 = 0$; $\therefore$ $m^{n} - 2x + \frac{1}{m^{n}} = 0$; also $p^{n} = \frac{1}{m^{n}}$; $\therefore$ $m^{n} + p^{n} = 2x$; hence $pA + mB = \frac{1}{2n} \times \frac{2 - 2x^2}{1 - 2x^2 + x^4} = \frac{1}{n} \times \frac{1 - x^2}{1 - x^2} = \frac{1}{n}$. Again, $A = \frac{1}{2n} \times \frac{m}{1 - xm^{n}}$, and $B = \frac{1}{2n} \times \frac{p}{1 - xp^{n}}$; $\therefore$ $A + B = \frac{1}{2n} \times : \frac{m}{1 - xm^{n}} + \frac{p}{1 - xp^{n}} =$ (since $mp = 1$) $\frac{1}{2n} \times : \frac{m + p - x . \overline{m^{n-1} + p^{n-1}}}{1 - x . \overline{m^{n} + p^{n}} + x^2 . m^{n} p^{n}}$. Now $m^{n} + p^{n} = 2x$, where $x$ is the cosine of an arc, which is to the arc whose cosine is $a$ :: $n$ : 1; for the same reason $m^{n-1} + p^{n-1} = 2e$, where $e$ is the cosine of an arc, which is to the arc whose cosine is $a$ :: $n - 1$ : 1; therefore, since $m + p = 2a$, $A + B = \frac{1}{2n} \times \frac{2a - 2xe}{1 - 2x^2 + x^2} = \frac{1}{2n} \times \frac{2a - 2xe}{1 - x^2}$. Hence $\frac{A}{m - v} + \frac{B}{p - v} =$

$\dfrac{\frac{1}{n}-\frac{a-xe}{n-nx^2}\times v}{1-2av+v^2}$; therefore $\dfrac{1}{v^{2n}-2xv^n+1}=\dfrac{\frac{1}{n}-\frac{a-xe}{n-nx^2}\times v}{1-2av+v^2}+\dfrac{\frac{1}{n}-\frac{b-xf}{n-nx^2}\times v}{1-2bv+v^2}+$ &c.; where $f$ is found from $b$ in the same manner that $e$ is obtained from $a$.

Cor. If $x$ be negative, the quantity to be resolved is $\dfrac{1}{v^{2n}+2xv^n+1}$; it $=\dfrac{\frac{1}{n}-\frac{a+xe}{n-nx^2}\times v}{1-2av+v^2}+\dfrac{\frac{1}{n}-\frac{b+xf}{n-nx^2}\times v}{1-2bv+v^2}+$ &c.

In the same way $\dfrac{1}{1\pm v^n}$ may be resolved; or it may be done by the following method:

## Prob. 78.

To resolve $\dfrac{1}{1-v^n}$ into its quadratic divisors, where $n$ is even.

By Cor. 1. Prob. 76, $1-v^n=\overline{1-v^2}\times\overline{1-2av+v^2}\times\overline{1-2bv+v^2}\times$ &c. to $\frac{n}{2}$ terms; therefore their hyperbolic logarithms are equal, or hyp. log. $\overline{1-v^n}=$ hyp. log. $\overline{1-v^2}+$ hyp. log. $\overline{1-2av+v^2}+$ hyp. log. $\overline{1-2bv+v^2}+$ &c.; therefore the fluxions on each side are equal; or,

$$\frac{-nv^{n-1}\dot{v}}{1-v^n}=\frac{-2v\dot{v}}{1-v^2}+\frac{-2a\dot{v}+2v\dot{v}}{1-2av+v^2}+\frac{-2b\dot{v}+2v\dot{v}}{1-2bv+v^2},\text{ to }\frac{n}{2}\text{ terms.}$$

Divide by $\frac{\dot{v}}{v}$, and

$$\frac{-nv^n}{1-v^n}=\frac{-2v^2}{1-v^2}+\frac{-2av+2v^2}{1-2av+v^2}+\frac{-2bv+2v^2}{1-2bv+v^2}\left(\frac{n}{2}\right).$$

Subtract each side from $n$, viz. the first from $n$, and each term on the other side from 2, so that the whole may be subtracted from $\frac{2\times n}{2}$ or $n$; hence,

$$\frac{n}{1-v^n} = \frac{2}{1-v^2} + \frac{2-2av}{1-2av+v^2} + \frac{2-2bv}{1-2bv+v^2} + \&c. \left(\frac{n}{2}\right).$$

$$\therefore \frac{1}{1-v^n} = \frac{\frac{2}{n}}{1-v^2} + \frac{\frac{2}{n}-\frac{2av}{n}}{1-2bv+v^2} + \frac{\frac{2}{n}-\frac{2bv}{n}}{1-2bv+v^2} + \&c. \text{ to}$$

$\frac{n}{2}$ terms.

COR. In the same manner $\frac{1}{r^n - v^n}$ may be resolved, where $n$ is an even number.

## PROB. 79.

To resolve $\frac{1}{1-v^n}$ into its quadratic divisors, where $n$ is odd.

By Cor. 2. Prob. 76, in this case $1-v^n = \overline{1-v} \times \overline{1-2av+v^2} \times \overline{1-2bv+v^2} \times$ &c. to $\frac{n+1}{2}$ terms; ∴ hyp. log. $1-v^n =$ hyp. log. $\overline{1-v}$ + hyp. log. $\overline{1-2av+v^2}$ + hyp. log. $\overline{1-2bv+v^2}$ + &c. $\left(\frac{n+1}{2}\right)$; therefore their fluxions are equal, or

$$\frac{-nv^{n-1}\dot{v}}{1-v^n} = \frac{-\dot{v}}{1-v} + \frac{-2a\dot{v}+2v\dot{v}}{1-2av+v^2} + \frac{-2b\dot{v}+2v\dot{v}}{1-2bv+v^2} + \&c. \left(\frac{n+1}{2}\right).$$

Divide each side by $\frac{\dot{v}}{v}$; then,

$$\frac{-nv^n}{1-v^n} = \frac{-v}{1-v} + \frac{-2av+2v^2}{1-2av+v^2} + \frac{-2bv+2v^2}{1-2bv+v^2} \left(\frac{n+1}{2}\right),$$

Subtract each side from $n$; viz. the first from $n$, the first term of the other side from 1, and the remaining $\frac{n-1}{2}$ terms each from 2; so that the whole may be subtracted from $1 + \frac{n-1}{2} \times 2$, or $n$; and the result is

$$\frac{n}{1-v^n} = \frac{1}{1-v} + \frac{2-2av}{1-2av+v^2} + \frac{2-2bv}{1-2bv+v^2} + \&c. \left(\frac{n+1}{2}\right);$$

$$\therefore \frac{1}{1-v^n} = \frac{\frac{1}{n}}{1-v} + \frac{\frac{2}{n}-\frac{2av}{n}}{1-2av+v^2} + \frac{\frac{2}{n}-\frac{2bv}{n}}{1-2bv+v^2} + \&c. \left(\frac{n+1}{2}\right).$$

In the same manner $\frac{1}{r^n - v^n}$ may be resolved, where $n$ is an odd number.

## Prob. 80.

To resolve $\frac{1}{1+v^n}$ into its quadratic divisors, where $n$ is even.

By Cor. 3. Prob. 76, in this case,

$$1^n + v^n = \overline{1-2av+v^2} \times \overline{1-2bv+v^2} \times \overline{1-2cv+v^2} \times \&c. \ \frac{n}{2} \text{ terms};$$

then, by the same process as in Prob. 78, we get,

$$\frac{1}{1+v^n} = \frac{\frac{2}{n}-\frac{2av}{n}}{1-2av+v^2} + \frac{\frac{2}{n}-\frac{2bv}{n}}{1-2bv+v^2} + \&c. \text{ to } \frac{n}{2} \text{ terms.}$$

## Prob. 81.

To resolve $\frac{1}{1+v^n}$ into its quadratic divisors, where $n$ is odd.

By Prob. 76. Cor. 4, in this case,

$$1^n + v^n = \overline{1+v} \times \overline{1-2av+v^2} \times \overline{1-2bv+v^2} \times \&c. \ \frac{n+1}{2} \text{ terms};$$

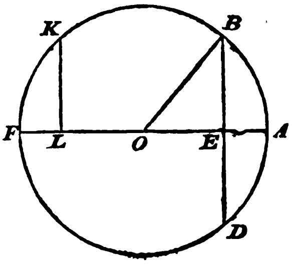
then, by the same process as in Prob. 79, we get,

$$\frac{1}{1+v^n} = \frac{\frac{1}{n}}{1+v} + \frac{\frac{2}{n}-\frac{2av}{n}}{1-2av+v^2} + \frac{\frac{2}{n}-\frac{2bv}{n}}{1-2bv+v^2} + \&c. \ \frac{n+1}{2} \text{ terms.}$$

In the same manner $\frac{1}{r^n + v^n}$ may be resolved.

## Prob. 82.

Given the sine of an arc, to find the sine of $n$ times that arc.



If $AB=z$, $AK=nz$, $OE=y$, and $OL$ the cosine of $AK$ $=x$; then, by Prob. 75, the radius being 1, we have $x+\sqrt{x^2-1}=\overline{y+\sqrt{y^2-1}}|^n$, or $x+\sqrt{x^2-1}=y^n+ny^{n-1}\times\sqrt{y^2-1}+n.\overline{\frac{n-1}{2}}\,y^{n-2}\times\overline{y^2-1}+n.\overline{\frac{n-1}{2}}.\overline{\frac{n-2}{3}}\times y^{n-3}\times\sqrt{y^2-1}\times\overline{y^2-1}+$ &c. This equation consists of quantities, part of which is possible and part impossible; hence the possible and impossible parts on each side are respectively equal. Therefore, taking the impossible parts, we have $\sqrt{x^2-1}=ny^{n-1}\times\sqrt{y^2-1}+n.\overline{\frac{n-1}{2}}.\overline{\frac{n-2}{3}}\times y^{n-3}\times\sqrt{y^2-1}\times\overline{y^2-1}+$ &c. Multiply both sides by $\sqrt{-1}$; then $\sqrt{1-x^2}=ny^{n-1}\times\sqrt{1-y^2}+n.\overline{\frac{n-1}{2}}\times\overline{\frac{n-2}{3}}\times y^{n-3}\times\sqrt{1-y^2}\times\overline{y^2-1}+$ &c. $=$ (if $s=\sqrt{1-y^2}$ or $-s^2=\overline{y^2-1}$) $ny^{n-1}s-n.\overline{\frac{n-1}{2}}.\overline{\frac{n-2}{3}}\,y^{n-3}s^3+$ &c., which is the sine of $AK$, $n$ times the arc $AB$.

### Prob. 83.

Given the sine of an arc, to find the sine of an $n^{th}$ part of that arc.

By the same notation, since $x+\sqrt{x^2-1}=\overline{y+\sqrt{y^2-1}}|^n$; $\therefore y+\sqrt{y^2-1}=\overline{x+\sqrt{x^2-1}}|^{\frac{1}{n}}=x^{\frac{1}{n}}+\frac{x^{\frac{1-n}{n}}\times\sqrt{x^2-1}}{n}+\frac{1}{n}\times\overline{\frac{1-n}{2n}}\times x^{\frac{1-2n}{n}}\times\overline{x^2-1}+\frac{1}{n}\times\overline{\frac{1-n}{2n}}\times\overline{\frac{1-2n}{3n}}\times x^{\frac{1-3n}{n}}\sqrt{x^2-1}\times\overline{x^2-1}+$ &c. Make the impossible parts equal; multiply by $\sqrt{-1}$, and let $S=\sqrt{1-x^2}$, the sine of $AK$; then $\sqrt{1-y^2}=\frac{x^{\frac{1-n}{n}}S}{n}-\frac{1.\overline{1-n}.\overline{1-2n}}{n.2n.3n}\times x^{\frac{1-3n}{n}}S^3+$&c., the sine of $AB$.

### PROB. 84.

Given the sine or cosine of an arc, to find the cosine of $n$ times that arc; or given the cosine of $AB$, to find that of $AK$.

Assume the possible parts of the equation in Prob. 82. equal; then $x = y^n + n . \overline{\frac{n-1}{2}} . y^{n-2} \times \overline{y^2 - 1} + \&c. = y^n - n . \overline{\frac{n-1}{2}} y^{n-2} s^2 + \&c.$, the cosine of $AK$.

### PROB. 85.

Given the sine or cosine of an arc, to find the cosine of an $n^{th}$ part of that arc; or given the cosine of $AK$, to find that of $AB$.

Make the possible parts of the equation in Prob. 83. equal; then $y = x^{\frac{1}{n}} + \frac{1}{n} \times \overline{\frac{1-n}{2n}} \times x^{\frac{1-2n}{n}} \times \overline{x^2 - 1} + \&c.$, or

$$y = x^{\frac{1}{n}} - \overline{\frac{1-n}{n . 2n}} \times x^{\frac{1-2n}{n}} S^2 + \&c.,\text{ the cosine of } AB.$$

### PROB. 86.

Given the sine or cosine, and therefore the tangent of an arc, to find the tangent of $n$ times that arc.

Let $t =$ the tangent of $AB$; then, by trigonometry, $t = \frac{s}{y}$ to radius 1; therefore the tangent of $AK = \frac{S, AK}{\text{cos. } AK} =$

(by Probs. 82. and 84.) $\dfrac{ny^{n-1}s - n . \overline{\frac{n-1}{2}} . \overline{\frac{n-2}{3}} y^{n-3} s^3 + \&c.}{y^n - n . \overline{\frac{n-1}{2}} . y^{n-2} s^2 + \&c.}$

$=$ (dividing both the numerator and denominator by $y^n$)

$$\frac{\frac{ns}{y}-n.\frac{\overline{n-1}}{2}.\frac{\overline{n-2}}{3}.\frac{s^3}{y^3}+\&c.}{1-n.\frac{\overline{n-1}}{2}.\frac{s^2}{y^2}-\&c.}=\frac{nt-n.\frac{\overline{n-1}}{2}.\frac{\overline{n-2}}{3}t^3+\&c.}{1-n.\frac{\overline{n-1}}{2}.t^2+\&c.}$$

### Prob. 87.

Given the same, to find the tangent of an $n^{th}$ part of the arc.

In the same manner as before, by Probs. 83. and 85, the

$$\text{tangent of } AB=\frac{\text{sine of } AB}{\text{cos. } AB}=\frac{\frac{x^{\frac{1-n}{n}}S}{n}-\frac{1.\overline{1-n}.\overline{1-2n}}{n.2n.3n}.x^{\frac{1-3n}{n}}S^3+\&c.}{x^{\frac{1}{n}}-\frac{\overline{1-n}}{n.2n}\times x^{\frac{1-2n}{n}}S^2+\&c.}$$

$$=\text{(dividing by } x^{\frac{1}{n}}\text{)}\ \frac{\frac{S}{nx}-\frac{1.\overline{1-n}.\overline{1-2n}}{n.2n.3n}\times\frac{S^3}{x^3}+\&c.}{1-\frac{\overline{1-n}}{n.2n}\times\frac{S^2}{x^2}+\&c.}\ \text{(if } T \text{ be}$$

$$\text{the tangent of } AK\text{)}=\frac{\frac{T}{n}-\frac{1.\overline{1-n}.\overline{1-2n}}{n.2n.3n}\times T^3+\&c.}{1-\frac{\overline{1-n}}{n-2n}\times T^2+\&c.}.$$

Cor. And since the secant of an arc $=\frac{r^2}{\text{cosine}}$, if the sine or cosine, and therefore the secant, of an arc be given, we can find the secant of $n$ times that arc or of an $n^{th}$ part of it.

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## SECTION IV.

### Prob. 88.

Required the quantity of matter in a sphere, whose density varies as the $n^{th}$ power of the distance from the center.

Let $d$ = the density of the sphere at the surface, $r$ = the radius, $x$ = any variable distance from the center, $p$ = the area of a circle whose radius is 1. Then the surface of a sphere, whose radius is $x$, $=4px^2$; and since $r^n : x^n :: d$ : density at the distance $x$, this density $= \frac{dx^n}{r^n}$; therefore the fluxion of the quantity of matter $= 4px^2\dot{x} \times \frac{dx^n}{r^n}$, and the content $= \frac{4pdx^{n+3}}{\overline{n+3}\,.\,r^n}$; or the quantity of matter in the whole sphere $= \frac{4pdr^3}{n+3}$.

Cor. 1. The quantities of matter in two spheres, whose densities are equal at the surface, and vary according to the same law of the distance from the center, are as the cubes of the radii.

Cor. 2. If $n=0$, or the density is constant, the content $= \frac{4pdr^3}{3}$.

Cor. 3. If two equal spheres have the same density at the surface, and the density of one be constant, whilst that of the other varies as the distance from the center,

The content of the former : that of the latter $:: \frac{4pdr^3}{3} : \frac{4pdr^3}{4}$

$:: 4 : 3$.

Cor. 4. If the density of a globe vary as the $n^{th}$ power of the distance from the center, and the density of its circumscribing cylinder be uniformly the same with that of the surface of the globe, The content of the globe : content of the cylinder $:: \frac{4pdr^3}{n+3} : 2pdr^3$

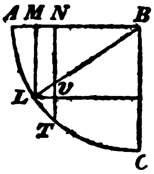
$:: 2 : n+3$.

Cor. 5. If the quantity of matter in a sphere be equal to that of its circumscribing cylinder, the density of the cylinder being uniformly the same with that of the surface of the sphere, the law of the density may be determined.

For $\frac{4pdr^{3n}}{n+3} = 2pdr^3$; $\therefore\ 2 = n+3$, and $n = -1$, or the density varies inversely as the distance from the center.

## PROB. 89.

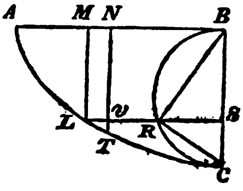
If a body descend by gravity down the quadrant of a circle *AC*, the radius *AB* being parallel to the horizon; to find where the velocity in the direction *BC* is a maximum.



Let $ML = y$, $BL = r$, $AL = z$; then the velocity at $L \propto \sqrt{ML} \propto \sqrt{y}$; and the velocity in the curve : velocity in the direction $BC$ :: $LT : vT$, ($TN$ being drawn parallel and near to $ML$, and $Lv$ perpendicular,) or :: $LB : BM :: r : \sqrt{r^2 - y^2}$; $\therefore$ the velocity in the direction $BC$ $\propto \frac{\sqrt{r^2y - y^3}}{r}$, a maximum; hence $r^2y - y^3$ is a maximum; its fluxion $= 0$, and $y = \frac{r}{\sqrt{3}}$.

## PROB. 90.

To find the same, if a body descend down the arc of a cycloid.



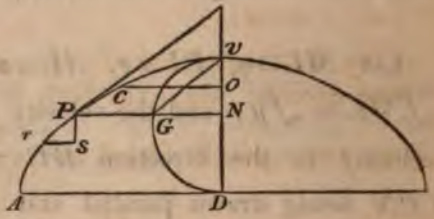
The same assumption being made, velocity in the direction $LT$ : velocity in the direction $BC$ :: $LT : vT$; by similar triangles, :: $RC : CS$ :: $BC : CR$ :: $\sqrt{BC} : \sqrt{CS}$; that is, :: $\sqrt{a} : \sqrt{a-y}$; hence the velocity in the direction $BC$ varies as $\frac{\sqrt{ay - y^2}}{\sqrt{a}}$, a maximum; therefore the fluxion of $ay - y^2 = 0$, or $2y\dot{y} = a\dot{y}$, and $y = \frac{a}{2}$.

### PROB. 91.

To find where a body, moving on the convex side of a cycloid $VPA$, will fly off, its motion being supposed to begin from the point $C$.

After leaving the cycloid, the body will describe a parabola, and its horizontal velocity will therefore be constant; hence the fluxion of the velocity in a horizontal direction, at the point where the body leaves the curve, must = 0. Suppose that point to be $P$; let $VD = a$, $VO = d$, $VN = x$; then $VG = \sqrt{ax}$, and $GN = \sqrt{ax - x^2}$. Now the velocity at $P$ = that acquired in falling through $ON$ or $x - d$, $= \sqrt{4m \times \overline{x-d}}$, where $m = 16\frac{1}{12}$ feet. And this velocity $(\sqrt{4m \times \overline{x-d}})$ : horizontal velocity :: $Pr : sr$ :: $VG : GN$

$$:: \sqrt{ax} : \sqrt{ax - x^2};$$



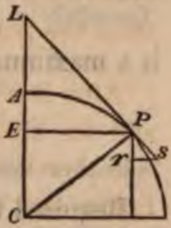
$\therefore$ the horizontal velocity $= \dfrac{\sqrt{4m \times \overline{x-d}} \times \sqrt{ax - x^2}}{\sqrt{ax}}$, or $\propto \sqrt{x - d} \times \sqrt{a - x}$, whose fluxion = 0; $\therefore$ the fluxion of $\overline{x - d} \times \overline{a - x} = 0$, or $\dot{x} \times \overline{a - x} - \dot{x} \times \overline{x - d} = 0$; $\therefore$ $a - x = x - d$, and $x = \dfrac{a + d}{2}$.

COR. If the body be just put in motion from $V$, $d = 0$, and $x = \frac{a}{2}$; or the body will leave the cycloid at a point determined by drawing the ordinate $NP$ from the center of the generating circle.

### PROB. 92.

A body begins to roll from $A$ on the quadrant of a circle with the velocity acquired by falling through $LA$; to determine the point where it will fly off from the curve.

Let $LA=d$, $AE=x$, $AC=a$; then the velocity in the curve at $P$, as before, $=\sqrt{4m \times \overline{d+x}}$; and $\sqrt{4m \times \overline{d+x}}$ : horizontal velocity :: $Ps : rs$ :: $PC : CE$,
:: $a : a-x$;
$\therefore$ the horizontal velocity $\propto \sqrt{d+x} \times \overline{a-x}$, whose fluxion, when the body leaves the curve, $=0$; and $x$ will be found $=\dfrac{a-2d}{3}$.



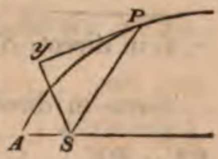
COR. 1. If $AL=\dfrac{AC}{2}$, $AE=0$, and the body moves in the direction of a tangent at $A$. If $AL$ be greater than $\frac{1}{2}AC$, $x$ is negative, or the Problem impossible.

COR. 2. If $AL=0$, $AE=\dfrac{AC}{3}$; but in this case the body must be supposed to be just set in motion at $A$, or it would have no tendency to roll on the arc, a tangent at $A$ being parallel to the horizon.

### PROB. 93.

Required that point in a parabola, where the linear velocity increases or decreases the fastest.

Let $AP$ be the parabola, $S$ the focus, $PY$ a tangent at $P$; join $SP$, and draw $SY$ perpendicular to the tangent. Now the increment of the velocity varies as the force × the increment of the time; therefore, dato tempore, the increment of velocity varies as the force, and is a maximum when the tangential force is the greatest.

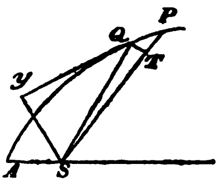


But the force in the direction $PS$ : force in direction $PY$ :: $SP$ : $PY$; therefore, since the force in the direction $PS \propto \frac{1}{SP^2}$, the tangential force in the direction $PY \propto \frac{PY}{SP^3}$, which in this case is a maximum. Let $SA=a$, $SP=x$; then $PY=\sqrt{x^2-ax}$; $\therefore$ $\frac{\sqrt{x^2-ax}}{x^3}$, or $\frac{x^2-ax}{x^6}$, or $\frac{x-a}{x^5}$, is a maximum; hence $x^5\dot{x}-5x^5\dot{x}+5ax^4\dot{x}=0$; $\therefore$ $x=\frac{5a}{4}$.

## Prob. 94.

Required that point in a parabola, where the angular velocity increases or decreases the fastest.

Draw $SQ$ indefinitely near $SP$, and $QT$ perpendicular to it; the angular velocity varies as $\frac{QT}{SP}$; therefore the angular velocity itself is a maximum, when $\frac{QT}{SP}$ is a maximum. But the increment of the angular velocity varies as the $\frac{\text{increment of } QT}{SP}$, and the increment of $QT$ varies as the force in the direction $QT$; therefore the increment of angular velocity is a maximum when $\frac{\text{force in direction } QT}{SP}$ is a maximum.



Now force in direction $SP$ : force in direction $PY$ :: $SP$ : $PY$, and f. in direct. $PY$ : f. in direct. $QT$ :: $PQ$ : $QT$ or :: $SP$ : $SY$;

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$\therefore$ f. in direct. $SP$ : f. in direct. $QT$ :: $SP^2$ : $SY \times PY$;

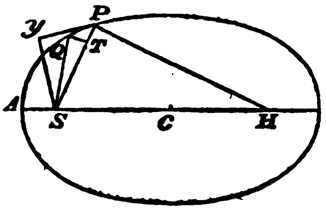
$\therefore$ force in direction $QT \propto \frac{SY \times PY}{SP^4}$; and $\frac{\text{this force}}{SP} \propto \frac{SY \times PY}{SP^5}$, which is a maximum.

Let $SA = a$, $SP = x$; then $SY = \sqrt{ax}$, and $PY = \sqrt{x^2 - ax}$; $\therefore \frac{\sqrt{ax} \times \sqrt{x^2 - ax}}{x^5}$, or $\frac{\sqrt{x-a}}{x^4}$, or $\frac{x-a}{x^8}$, is a maximum; hence $x^8\dot{x} - 8x^8\dot{x} + 8ax^7\dot{x} = 0$, and $x = \frac{8a}{7}$.

The same principle might be adopted in the two following Propositions; but it is more convenient to use a different method.

### Prob. 95.

Required that point in an ellipse, where the linear velocity increases or decreases the fastest.



Take $P$ some point in the ellipse, whose foci are $S$, $H$; join $SP$, $HP$, and draw $SY$ perpendicular to a tangent at $P$. Let $SP = x$, $HP = v$, the semi-major axis $= a$, and semi-minor $= b$. Now the linear velocity $\propto \frac{1}{SY} \propto \sqrt{\frac{v}{x}}$, from the nature of the ellipse; $\therefore$ the fluxion of the linear velocity $\propto \frac{\dot{v}x - \dot{x}v}{x\sqrt{vx}}$; but $x + v = 2a$, or $\dot{v} = -\dot{x}$; $\therefore$ the fluxion of the linear velocity $\propto \frac{-\dot{x} \times \overline{x+v}}{x\sqrt{vx}} \propto \frac{\dot{x}}{x\sqrt{vx}}$. But the area $SPQ$ described, dato tempore, round $S$ is constant in the same curve (Newton, Prop. 1.); therefore $SP \times QT$ is equal to a constant quantity; and
$PT : QT :: PY : SY$, or
$\dot{x} : QT :: \sqrt{\frac{x^2v - b^2x}{v}} : b \times \sqrt{\frac{x}{v}}$, by the nature of the ellipse; $\therefore QT = \frac{b\dot{x}}{\sqrt{vx - b^2}}$; $\therefore SP \times QT \propto \frac{x\dot{x}}{\sqrt{vx - b^2}}$, which is constant, or $\propto 1$; and $\dot{x} \propto \frac{\sqrt{vx - b^2}}{x}$; hence the

expression $\frac{\dot{x}}{x\sqrt{vx}} \propto \frac{\sqrt{vx-b^2}}{x^2 \times v^{\frac{1}{2}}x^{\frac{1}{2}}}$, which is a maximum; that is, $\frac{vx-b^2}{vx^5}$ is a maximum, or $\frac{1}{x^4} - \frac{b^2}{vx^5}$; therefore, taking the fluxion, $\frac{-4x^3\dot{x}}{x^8} + \frac{b^2 \times \overline{5x^4\dot{x}v + x^5\dot{v}}}{v^2x^{10}} = 0$; therefore $\frac{-4v^2x^5\dot{x} + 5b^2x^4v\dot{x} + b^2x^5\dot{v}}{v^2x^{10}} = 0$; omit the denominator, and for $\dot{v}$ write $-\dot{x}$; then $5b^2x^4v - b^2x^5 - 4v^2x^5 = 0$; that is, $5b^2v - b^2x - 4v^2x = 0$. For $x$ write $2a-v$;

then $5b^2v + b^2v - 2ab^2 + 4v^3 - 8av^2 = 0$,
or $4v^3 + 6b^2v - 8av^2 - 2ab^2 = 0$,
or $2v^3 - 4av^2 + 3b^2v - ab^2 = 0$,

a cubic equation, in which there will be only one possible value of $v$.

## Prob. 96.

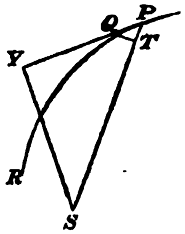
Required that point in an ellipse, where the angular velocity increases or decreases the fastest. (See preceding Fig.)

Draw $SQ$ indefinitely near, and $QT$ perpendicular to $SP$. The angular velocity varies as $\frac{\text{the area described dato tempore}}{\overline{\text{dist.}}|^2}$ $\propto \frac{1}{SP^2}$, since the area dato tempore is constant, $\propto \frac{1}{x^2}$; hence the fluxion of the angular velocity $\propto \frac{x\dot{x}}{x^4} \propto \frac{\dot{x}}{x^3}$. But $\dot{x} \propto \frac{\sqrt{vx-b^2}}{x}$; $\therefore \frac{\dot{x}}{x^3} \propto \frac{\sqrt{vx-b^2}}{x^4}$, which is a maximum, and its fluxion $=0$; that is, $\frac{\overline{v\dot{x}+x\dot{v}} \times x^4}{2 \times \sqrt{vx-b^2}} - 4x^3\dot{x} \times \sqrt{vx-b^2}$ $=0$. But $\dot{x}+\dot{v}=0$, and $\dot{x} = -\dot{v}$; $\therefore \frac{\overline{v-x} \times \dot{x} \times x^4}{2\sqrt{vx-b^2}} = 4x^3\dot{x} \times \sqrt{vx-b^2}$; hence $-x^2 = 7vx - 8b^2$. Now for $v$ write $2a-x$, and $x$ will be found $= \frac{7a - \sqrt{49a^2 - 48b^2}}{6}$. The other value of $x$ is of no use; for it is a quantity greater than $x$ ever can be in the ellipse.

### Prob. 97.

To find the point, where the paracentric velocity of a body revolving in any curve is a maximum.

Let $PR$ be the curve, $S$ the center of force, $PY$ a tangent at $P$, and $SY$ perpendicular to it. Take $PQ$ a small arc in the curve, and $QT$ perpendicular to $SP$. Then the velocity in the curve : paracentric velocity $:: QP : PT$ $:: SP : PY$;



$\therefore$ the paracentric vel. $= \frac{\text{vel. in curve} \times PY}{SP}$ $\propto \frac{PY}{SY \times SP}$ (for the velocity in the curve $\propto \frac{1}{SY}$) $\propto \sqrt{\frac{SP^2 - SY^2}{SP \times SY}}$. Hence, in the case required, this expression, or its square $\frac{SP^2 - SY^2}{SP^2 \times SY^2}$, is a maximum.

### Examples.

Ex. 1. Let the curve be a parabola.

Here if $SP = x$, $SY^2 = ax$, and $\frac{x^2 - ax}{ax \times x^2}$, or $\frac{x - a}{x^2}$, is a maximum; $\therefore x^2\dot{x} - 2x\dot{x} \times \overline{x - a} = 0$; $\therefore x = 2x - 2a$, and $x = 2a = \frac{1}{2}$ the latus rectum.

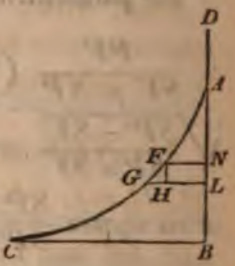
Ex. 2. Let the curve be an ellipse. (Fig. Prob. 95.)

If $SP = x$, $HP = v$, $AC = a$, and the semi-minor axis $= b$; then $SY^2 = \frac{b^2x}{v}$, and $SP^2 - SY^2 = x^2 - \frac{b^2x}{v} = \frac{vx^2 - b^2x}{v}$, and $SP^2$

$\times SY^2 = \frac{b^2x^3}{v}$; therefore $\frac{SP^2 - SY^2}{SP^2 \times SY^2} = \frac{vx^2 - b^2x}{b^2x^3} \propto \frac{vx - b^2}{x^2} \propto \frac{\overline{2a-x} \times x - b^2}{x^2} \propto \frac{2ax - x^2 - b^2}{x^2} \propto \frac{2ax - b^2}{x^2} - 1$, which is a maximum; and $2ax^2\dot{x} - 2x\dot{x} \times \overline{2ax - b^2} = 0$, or $ax = 2ax - b^2$; $\therefore\ x = \frac{b^2}{a} = \frac{1}{2}$ the latus rectum.

### Prob. 98.

To determine the nature of the curve, down which a body must descend after its fall through a given space by the action of a constant force, so that it may, in the direction of its first descent, describe equal spaces in equal times.



Let the space $DA$ through which the body falls $= a$, and suppose $AFC$ to be the curve required; take $FG$ a small arc, and on $DA$ produced draw $FN$, $GL$, $CB$ perpendiculars, and let $FH$ be parallel to $AB$. Take $AN = x$, $AF = z$, $FN = y$; then $FG = \dot{z}$, $FH = \dot{x}$, $HG = \dot{y}$. Now the velocity in the curve at $F =$ that through $DN$, and therefore varies as $\sqrt{a+x}$; and the velocity in the direction $FH$ is constant by supposition, and equal to that acquired down $DA$;

$$\therefore\ \dot{z} : \dot{x} :: \sqrt{a+x} : \sqrt{a}, \text{ or } \dot{z}^2 : \dot{x}^2 :: a+x : a;$$

that is, $\dot{x}^2 + \dot{y}^2 : \dot{x}^2 :: a+x : a$, and $\dot{y}^2 : \dot{x}^2 :: x : a$;

$$\therefore\ \dot{y} : \dot{x} :: x^{\frac{1}{2}} : a^{\frac{1}{2}}, \text{ and } a^{\frac{1}{2}}\dot{y} = x^{\frac{1}{2}}\dot{x};\ \therefore\ a^{\frac{1}{2}}y = \frac{2x^{\frac{3}{2}}}{3};$$

or $\frac{9ay^2}{4} = x^3$, and the curve is the semi-cubical parabola.

Cor. The latus rectum of this semi-cubical parabola $= \frac{9a}{4}$.

### PROB. 99.

A body describes a curve, whose equation is $a^{n-1}y=x^n$, by a force acting in parallel lines, and in the direction of the ordinates; required the variation of the velocity.

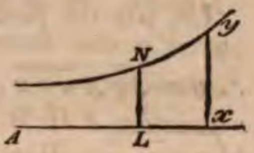
By Art. 161. $v^2 \propto F \times PV \propto -\ddot{y} \times \frac{\dot{z}^2}{-\ddot{y}}$ (Art. 103.) $\propto \dot{z}^2$.

Now $\dot{y}=\frac{nx^{n-1}\dot{x}}{a^{n-1}}$; $\therefore\ \dot{y}^2=\frac{n^2}{a^{2n-2}}\times x^{2n-2}\dot{x}^2$, and $\dot{z}^2=\dot{x}^2+\dot{y}^2=\frac{\overline{n^2x^{2n-2}+a^{2n-2}}\times\dot{x}^2}{a^{2n-2}}$; $\therefore\ v \propto \sqrt{n^2x^{2n-2}+a^{2n-2}}$.

### PROB. 100.

To determine the curve, in which a body revolving by a force, which acts in lines perpendicular to the axis, shall approach to or leave the axis with a velocity always proportional to the ordinates.

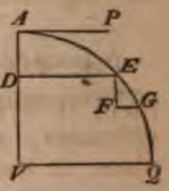
Let $NY$ be the curve, $Ax$ the axis, $LN$ and $xy$ ordinates; let $AL=x$, $LN=y$. Then $\dot{x}$ is constant, and by the Problem $\dot{y} \propto y$; $\therefore\ \frac{\dot{y}}{y}$ is constant, or $\frac{\dot{y}}{y} \propto \dot{x}$, a property of the logarithmic curve. (Art. 132.)



### PROB. 101.

A body is projected with a given velocity from $A$, in a direction $AP$ parallel to the plane $VQ$; to find the curve described by the body, the attractive force of each particle being supposed constant.

Let $AQ$ be the curve described; assume $AV$ perpendicular to the plane $=a$, $VD$ a variable distance $=x$, $DE$ an ordinate $=y$, $EF=-\dot{x}$, $FG=\dot{y}$, and let the uniform velocity in the direction $AP=b$. Then, if $v$ represent the velocity at $E$ in the direction $AV$, $v\dot{v} \propto -F\dot{x}$; $\therefore v^2 \propto -x$ + corr., and $v^2 \propto a-x$; $\therefore v \propto \sqrt{a-x} = d^{\frac{1}{2}} \times \sqrt{a-x}$. But the velocities are as the spaces described uniformly dato tempore; $\therefore$ $-\dot{x} : \dot{y} :: d^{\frac{1}{2}} \times \sqrt{a-x} : b$; therefore $\dot{y} = -\dfrac{b\dot{x}}{d^{\frac{1}{2}} \times \sqrt{a-x}}$, and $y = \dfrac{2b}{d^{\frac{1}{2}}} \times \sqrt{a-x}$; that is, the curve $AQ$ is a parabola, whose vertex is $A$, and principal latus rectum $= \dfrac{4b^2}{d}$.



Cor. This corresponds with the motion of a body projected at the earth's surface, where $d=4m$, and the latus rectum $= \dfrac{b^2}{m}$, $m$ being $16\frac{1}{12}$ feet.

### Prob. 102.

Conversely, a body projected in a direction $AP$ parallel to the plane $VQ$, whose particles attract it according to a certain law, describes a parabola $AQ$; required the law of the force.

Let $p =$ the parameter of the parabola; then $p \times \overline{a-x} = y^2$; $\therefore y = \sqrt{p \times \overline{a-x}}$, and $\dot{y} = \dfrac{-\frac{1}{2}p\dot{x}}{\sqrt{p \times \overline{a-x}}}$; $\therefore \dot{y} : -\dot{x} :: \frac{1}{2}p : \sqrt{p \times \overline{a-x}}$. But $\dot{y} : -\dot{x}$ :: the uniform velocity ($b$) in the direction $AP$ : the velocity ($V$) at $E$;

$$\therefore b : v :: \tfrac{1}{2}p : \sqrt{p \times \overline{a-x}};$$

hence $v \propto \sqrt{a-x}$, and $v^2 \propto a-x$; $\therefore v\dot{v} \propto -\dot{x}$. But $v\dot{v} \propto -F\dot{x}$, where $F$ is the accelerating force; $\therefore -F\dot{x} \propto -\dot{x}$, and $F \propto 1$, or is constant.

### Prob. 103.

A body is projected with a given velocity from $A$ in a direction $AP$ parallel to the plane $VQ$; to find the curve described by the body, the attractive force of each particle varying inversely as the $\overline{\text{dist.}}\vert^3$. (See preceding Fig.)

Let *AQ* be the curve described, and assume as before; then, if $v$ represent the velocity at any distance $x$ from the plane in the direction *AV*, $v\dot{v} \propto -F\dot{x} \propto \frac{-\dot{x}}{x^3}$; $\therefore v^2 \propto \frac{1}{x^2} - \frac{1}{a^2} \propto \frac{a^2-x^2}{a^2x^2}$, and $v \propto \frac{\sqrt{a^2-x^2}}{x}$. But the velocity $\propto$ space uniformly described dato tempore; $\therefore -\dot{x} : \dot{y} :: d \times \frac{\sqrt{a^2-x^2}}{x} : b$; the factor $d$ being assumed in the third term to make that term of the same dimensions with the fourth; hence $\dot{y} = -\frac{bx\dot{x}}{d \times \sqrt{a^2-x^2}}$, and $y = \frac{b}{d} \times \sqrt{a^2-x^2}$, which is the property of an ellipse; therefore the curve *AQ* is an ellipse, whose semi-major is $AV = a$, and semi-minor axis $VQ = \frac{ab}{d}$.

Cor. 1. If $b = d$, the curve is a circle.

Cor. 2. If the force be repulsive, it may be shewn in the same manner, that the curve is an hyperbola, whose center is *V*, and vertex *A*; for in this case $y = \frac{b}{d} \times \sqrt{x^2-a^2}$.

### Prob. 104.

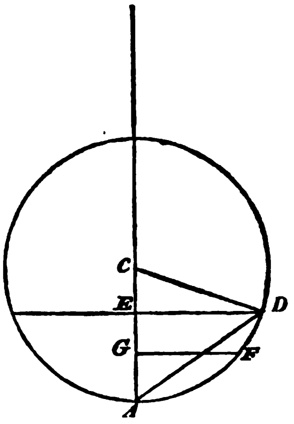
Conversely, a body projected in a direction *AP* parallel to the plane *VQ*, whose particles attract it according to a certain law, describes an ellipse *AQ*; required the law of the force.

Here if $c$ = the semi-minor, and $a$ the semi-major axis of the ellipse, $y = \frac{c}{a} \times \sqrt{a^2-x^2}$; $\therefore \dot{y} = \frac{c}{a} \times \frac{-x\dot{x}}{\sqrt{a^2-x^2}}$;

$\therefore \dot{y} : -\dot{x} :: cx : a \times \sqrt{a^2 - x^2}$. But $\dot{y} : -\dot{x}$ :: the uniform velocity in the direction $AP$ ($b$) : the velocity towards the plane at $E$ ($v$); $\therefore b : v :: cx : a \times \sqrt{a^2 - x^2}$; $\therefore v \propto \frac{\sqrt{a^2 - x^2}}{x}$, and $v^2 \propto \frac{a^2 - x^2}{x^2}$. Hence $v\dot{v} \propto -\frac{2a^2 x\dot{x}}{x^4} \propto -\frac{\dot{x}}{x^3}$; but if $F$ represent the accelerating force, $v\dot{v} \propto -F\dot{x}$; $\therefore -F\dot{x} \propto -\frac{\dot{x}}{x^3}$, and $F \propto \frac{1}{x^3}$.

## PROB. 105.

To find the time of vibration of a pendulum in the arc of a circle.



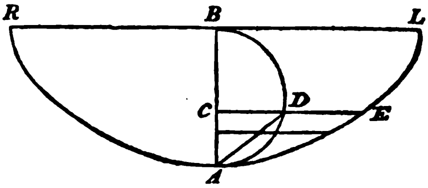
Let $D$ be the point from which the pendulum $CD$ begins its vibrations, and from any point $F$ in the arc $DA$ draw $FG$ parallel to the horizon, meeting the vertical line $CA$ in $G$. Let $CA = a$, $AG = x$, $AE = b$, and $AF = z$. Then $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - x^2}}$; and since the velocity at $F = \sqrt{4m \times \overline{b - x}}$, we have $\dot{T} = -\frac{\dot{z}}{V} = \frac{-a\dot{x}}{\sqrt{4m \times \overline{b-x}} \times \sqrt{2ax - x^2}} = \frac{-1}{\sqrt{4m}} \times \frac{a\dot{x}}{\sqrt{bx - x^2} \times \sqrt{2a - x}}$, whose fluent or $T$, found by expanding $\frac{1}{\sqrt{2a - x}}$, or $\overline{1 - \frac{x}{2a}}\big|^{-\frac{1}{2}}$ by the binomial theorem, $= +\frac{p\sqrt{a}}{\sqrt{8m}} \times : 1 + \frac{b}{2.2.2a} + \frac{3.3b^2}{2.2.4.4.\overline{2a}|^2}$ + &c., where $p =$ the circumference of a circle to the diameter 1; therefore the whole time of vibration, which is double of this,

$$= p\sqrt{\frac{a}{2m}} \times : 1 + \frac{b}{2.2.2a} + \frac{3.3b^2}{2.2.4.4.\overline{2a}|^2} + \&c.$$

**Cor.** The time of descent down the diameter, or the chord $AD$, $= \sqrt{\frac{2a}{m}}$; hence, if the arc in which the pendulum vibrates be very small, or $b$ evanescent, we have, time down arc or $\frac{1}{2}$ time of vibration : time down the chord $:: \frac{p}{2} \times \sqrt{\frac{a}{2m}} : \sqrt{\frac{2a}{m}} :: p : 4 ::$ the circumference of a circle : four times its diameter.

### Prob. 106.

To find the time in which a body will fall by gravity down the arc of a semi-cycloid.



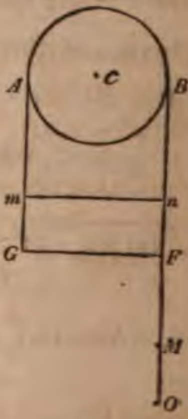
Let $AB = a$, $AC = x$, $AE = z$; then $AD = \sqrt{ax}$, and $AE = 2\sqrt{ax}$; therefore the fluxion of the arc $= a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$. Also $BC = a - x$; $\therefore$ if $m = 16\frac{1}{12}$ feet, velocity at $E = \sqrt{4m \times \overline{a-x}}$; $\therefore \dot{T} = \frac{-\dot{S}}{V} = \frac{-a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}}{\sqrt{4m \times \overline{a-x}}} = \frac{2\times}{\sqrt{4ma}} - \frac{\frac{1}{2}a\dot{x}}{\sqrt{ax-x^2}}$;

$\therefore$ (Art. 44.) $T = \frac{-2}{\sqrt{4ma}} \times$ a circular arc of radius $\frac{1}{2}a$, and versed sine $x$ + corr. when $x = a$, $T = 0$; $\therefore T = \frac{2}{\sqrt{4ma}} \times$ a circular arc of radius $\frac{1}{2}a$, and versed sine $a$; the remainder vanishing, when $x = 0$, $= \frac{2}{\sqrt{4ma}} \times \frac{1}{2}$ circumference $=$ time from $L$ to $A$; therefore the time from $L$ to $R = \frac{4}{\sqrt{4ma}}$

$\times \frac{1}{2}$ circumference $= \frac{2}{\sqrt{ma}} \times \frac{pa}{2}$, if $p = 3.14159$, &c. $=$ $p \times \sqrt{\frac{a}{m}}$. Now the time down the axis $= \sqrt{\frac{a}{m}}$; therefore time of vibration : time down axis $:: p\sqrt{\frac{a}{m}} : \sqrt{\frac{a}{m}} :: p : 1$ :: circumference of a circle : its diameter.

### PROB. 107.

If a chain of uniform density be hung over a fixed pulley, and the parts of the chain be unequal, the heavier will descend; given the length of the chain, and the parts of it on each side at the commencement of the motion, to find the time in which it will leave the pulley.



Let $AG$, $BM$ be the parts of the chain on each side of the pulley at the beginning of the motion. Draw $GF$ parallel to the horizon; then the moving force with which $M$ begins to descend is the part $FM$. Let $M$ descend to $O$, and take $Gm = MO$; the moving force then $=$ $nO = FM + 2MO$. Let $L =$ the whole length of the chain, and the force of gravity upon it $= 1$; then, if $FM = 2a$, and $MO = x$, $no = 2a + 2x$, and the accelerating force $= \frac{2a + 2x}{L}$. Let $z =$ the space through which a body would fall by gravity to acquire the velocity at $O$; then $\dot{z} = \frac{2a\dot{x} + 2x\dot{x}}{L}$, and $z = \frac{2ax + x^2}{L}$; $\therefore$ velocity at $O$ $(= \sqrt{4mz}$, if $m = 16\frac{1}{12}$ feet$) = \sqrt{\frac{4m}{L}} \times \sqrt{2ax + x^2}$. Hence $\dot{T} = \sqrt{\frac{L}{4m}} \times \frac{\dot{x}}{\sqrt{2ax + x^2}}$, and $T = \sqrt{\frac{L}{4m}} \times$ hyp. log. $a + x +$

$\sqrt{2ax+x^2}$ + corr. Let $T=0$, $x=0$, and corr. $=-\sqrt{\frac{L}{4m}}\times$ hyp. log. $a$; $\therefore T=\sqrt{\frac{L}{4m}}\times$ hyp. log. $\frac{a+x+\sqrt{2ax+x^2}}{a}$. Now when the chain quits the pulley, $x=\frac{1}{2}\,\overline{L-a}$; $\therefore$ the whole time $=\sqrt{\frac{L}{4m}}\times$ hyp. log. $\frac{\frac{1}{2}L+\sqrt{aL-2a^2+\frac{1}{4}\overline{L-a}|^2}}{a}$.

## Prob. 108.

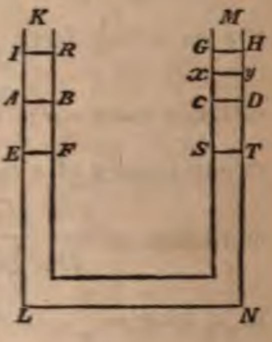
Suppose a weight, suspended by a cord passing over a fixed pulley, to be uniformly drawn up; required the number of vibrations which the weight would make before it reaches the pulley.

Let $a=$ the distance of the weight from the pulley at the beginning of its motion, $x=$ any variable distance ascended, and $v=$ the space through which it is uniformly drawn up in 1″; $m=16\frac{1}{12}$ feet; then the length of the pendulum at the distance $x=a-x$; therefore, time of oscillation : $\sqrt{\frac{a-x}{2m}}$ :: circumference of a circle : diameter; therefore the time of one oscillation $=\frac{C\times\sqrt{a-x}}{D\times\sqrt{2m}}$; hence the number of oscillations in a time $=\frac{\dot{x}}{v}$, or the fluxion of the number of oscillations, whilst the weight ascends through the space $x$, $=\frac{D\times\sqrt{2m}\times\dot{x}}{C\times V\times\sqrt{a-x}}$ $=\frac{\sqrt{2m}\times D}{C\times V}\times\frac{\dot{x}}{\sqrt{a-x}}$, whose fluent, when $x=a$, is $\frac{2a^{\frac{1}{2}}\times\sqrt{2m}\times D}{C\times V}=$ the number of vibrations made by the pendulum.

Cor. The number of vibrations, made by this pendulum, = twice the number made in the same time, viz. $\frac{a}{v}$, by a common pendulum, whose length is $a$.

## Prob. 109.

Let $KLNM$ represent a cylindrical tube, whose arms $KL$ and $NM$ are perpendicular, and $LN$ parallel to the horizon; suppose it to be partly filled with water, and the surface of the water, when quiescent, to coincide with $AB$ and $CD$; let $AB$ be depressed to $EF$, and $CD$ raised equally to $GH$; draw $IR$ in the other arm parallel to $LN$, and at the same height with $GH$; then, the pressure being removed, $GH$ will descend, and $EF$ ascend. Required the time in which $EF$ will rise to its greatest altitude $IR$.



Draw $ST$ in $MN$ at the same altitude with $EF$, and parallel to $LN$. Let $GC=a$; then $GS = 2a$, and the force with which the upper surface endeavours to descend = a column of the fluid, whose altitude is $2a$. Let the axis of the tube $=L$, and the force with which gravity would accelerate the fluid, if unconfined, $=1$; then the column $GS$ : the whole weight of the fluid :: $2a : L$; $\therefore$ the force of this column $=\frac{2a}{L}$. Let $GH$ descend to $xy$, and put $Gx = x$; then the accelerating force $=\frac{2a-2x}{L}$; hence, if $z$ be the space through which a body would fall by gravity to acquire the velocity at $xy$, $\dot{z}=\frac{2a\dot{x}-2x\dot{x}}{L}$, and $z=\frac{2ax-x^2}{L}$; $\therefore$ the velocity at $x=\sqrt{\frac{4m}{L}}\times\sqrt{2ax-x^2}$, if $m=16\frac{1}{12}$ feet.

Hence $\dot{T} = \frac{\dot{x}}{v} = \sqrt{\frac{L}{4m}} \times \frac{\dot{x}}{\sqrt{2ax - x^2}}$; and $T = \sqrt{\frac{L}{4ma^2}} \times$ a circular arc of radius $a$, and versed sine $x$, which needs no correction; $\therefore$ the whole time of ascent to $IR$, when $x = 2a$, is $\sqrt{\frac{L}{4ma^2}} \times \frac{1}{2}$ the circumference of a circle of radius $a = \sqrt{\frac{L}{4ma^2}} \times pa$ (if $p = 3.14159$, &c.) $= p \times \sqrt{\frac{L}{4m}}$.

Cor. 1. The time of one descent of the fluid from $GH$ to $ST$ is equal to the time in which a pendulum would vibrate, whose length is half the length of the tube's axis.

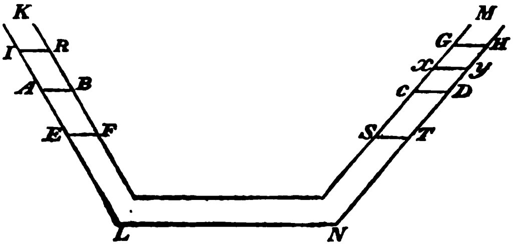
Cor. 2. Since $HD$ does not enter into the expression, whatever be the altitude of $GH$, the whole time of descent will be the same.

Cor. 3. The velocity of the descending surface is accelerated to $CD$, and then retarded to $ST$, where it $= 0$.

## Prob. 110.

Let the arms of the tube be inclined to the horizontal part $LN$ in any given angles, and let $AB$ be depressed as before to $EF$; to find the time, in which, if the pressure were removed, $EF$ would rise to its greatest altitude $IR$.

Let $AI$ or $AE$, which are equal to $HD$ and $DT$, $= a$; $p$ = the sine of the angle $KLN$, $q$ = the sine of the angle $MNL$, to a radius 1. Let $L$ = the length of the whole canal, and the force with which gravity would accelerate the fluid if unconfined = 1. Then the force, by which the column $HD$ endeavours to descend at first =



$\frac{HD \times \text{the sine of } HNL}{r} + \frac{AE \times \text{sine of } KLN}{r} = HD \times \overline{q+p}$ $= a \times \overline{q+p} =$ the weight of the two columns $HD$ and $AE$ in a perpendicular direction. Hence the accelerating force at $H = \frac{a \times \overline{q+p}}{L}$; and when the surface $GH$ has descended to $xy$, if $HY = x$, the accelerating force on $xy = a \times \frac{\overline{q+p}}{L} - x \times \frac{\overline{q+p}}{L} = \frac{\overline{a-x} \, . \, \overline{q+p}}{L}$; and if $z$ represent the space through which a body must fall freely by gravity to acquire the velocity at $xy$, $\dot{z} = \frac{\overline{a\dot{x} - x\dot{x}} \times \overline{q+p}}{L}$, and $z = \frac{\overline{2ax - x^2} \times \overline{q+p}}{2L}$; hence the velocity at $y = \sqrt{\frac{\overline{q+p}}{2L} \times 4m} \times \sqrt{2ax - x^2}$. Now $\dot{T} = \frac{\dot{x}}{v}$; $\therefore$ in this case, $T = \sqrt{\frac{2L}{4m \times \overline{q+p}}} \times f . \frac{\dot{x}}{\sqrt{2ax - x^2}}$, and $T = \sqrt{\frac{2L}{4ma^2 \times \overline{q+p}}} \times$ a circular arc of radius $a$, and versed sine $x$, which needs no correction. Let $x = 2a$; then the whole time $= \sqrt{\frac{2L}{4ma^2 \times \overline{q+p}}} \times \frac{1}{2}$ the circumference of a circle of radius $a = \sqrt{\frac{2L}{4ma^2 \times \overline{q+p}}} \times Pa$ (if $P = 3.14159$, &c.) $= P \times \sqrt{\frac{L}{2m \times \overline{q+p}}}$.

Cor. 1. If $KL$ and $MN$ are perpendicular to the horizon, or $p$ and $q$ each $= 1$, $T = P \times \sqrt{\frac{L}{4m}}$, as before.

Cor. 2. The time of one descent of the surface from $GH$ to $ST$, or ascent from $EF$ to $IR$, is equal to the time in which a pendulum would vibrate, whose length is $\frac{L}{p+q}$.

Cor. 3. When the angles at $L$ and $N$ each equal 30°, then $p$ and $q$ each $= \frac{1}{2}$; and the length of a pendulum, which would vibrate in the same time, $= L$.

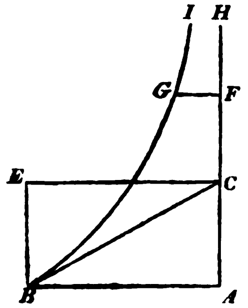
### Prob. 111.

The force of gravity being supposed constant, to find the density of the air at any altitude above the surface of the earth.

Let $r =$ the radius of the earth, $x =$ any variable distance from the center above the surface, $v =$ the density of the air at that point, the density at the surface being assumed $= 1$; $h =$ the height of an homogeneous atmosphere. Now since the compressive force of the air is proportional to its density, the fluxion of the compressive force is to the fluxion of the density as the compressive force to the density; and in this case, since the force is constant, the fluxion of the compressive force is as the density and the fluxion of the altitude; hence $h : 1 :: v\dot{x} : -\dot{v}$; $\therefore\ -h \times \frac{\dot{v}}{v} = \dot{x}$, and $-h \times$ hyp. log. $v +$ corr. $= x +$ corr., or $h \times$ hyp. log. $\frac{1}{v} = x - r$.

Cor. 1. If $x$, and therefore $x - r$, be assumed in arithmetic progression, the hyp. log. of $\frac{1}{v}$ is in arithmetic progression, and $\frac{1}{v}$ in geometric; that is, the densities of the air are in geometric progression.

Cor. 1. If $AB$ represent the density of the air at the earth's surface, and $AH$ be drawn perpendicular to $AB$, and any ordinates, as $FG$, be taken proportional to the density at $F$, the curve $BGI$ thus traced out is the logarithmic curve.



For $x - r \propto$ hyp. log. $\frac{1}{v}$, or $AF$ varies inversely as the logarithm of the density; hencc, if $AF$ increases in arithmetic progression, $FG$ decreases in geometric, a property of the logarithmic curve. (Art. 131.)

Cor. 2. Let $h =$ the height of a homogeneous atmosphere, in which the density throughout is the same with the density at $A$; the pressure at $A$ is the same, whether we take the homogeneous atmosphere, or that whose density is represented by the varying line $FG$. Hence $h \times AB =$ the area $ABIH$ $= AB \times AC$ (Art. 133. Prob. 3. Cor. 1.); $\therefore h = AC =$ the sub-tangent, or the modulus of this system of logarithms is the height of an homogeneous atmosphere. (Art. 132. Prob. 2. Cor. 2.) Cotes' *Harm. Mens.* Prop. 5. Scholium.

### Prob. 112.

The force of gravity varying inversely as the square of the distance from the center of the earth; to find the density of the air at any altitude above the surface.

The force of gravity at any distance $x = \frac{r^2}{x^2}$. Now the compressive force of the air varies as its density; and the fluxion of the compressive force is proportional to the force of gravity, the density and the fluxion of the altitude; hence, on the same assumption as in the last Problem,

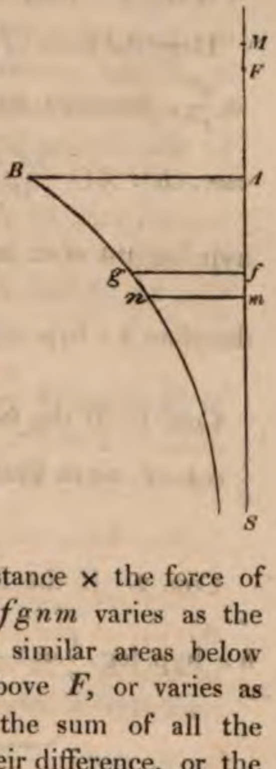
$$h : 1 :: \frac{v r^2 \dot{x}}{x^2} : -\dot{v}; \therefore h \times -\frac{\dot{v}}{v} = \frac{r^2 \dot{x}}{x^2},$$

and $-h \times$ hyp. log. $v = -\frac{r^2}{x}$ + corr.; that is, $h \times$ hyp. log. $\frac{1}{v}$ $= r - \frac{r^2}{x}$.

Cor. 1. If $x$ increase in musical progression, $\frac{r^2}{x}$ is in arithmetic; hence the hyp. log. of the densities decreases in arithmetic progression, and the densities themselves in geometric.

## COTES' SOLUTION. HARMONIA MENSURARUM.

### PROP. 5. SCHOLIUM.



Let $S$ be the center of the earth, $AB$ its surface. Assume $Sf : SA :: SA : SF$, and draw the ordinate $fg$ proportional to the density at $F$; the curve $Bgn$ traced out by the point $g$ will be the logarithmic curve, but in an inverted position. For let $AF$ be increased by the small line $FM$; take $Sm : SA :: SA : SM$, and assume $mn$, an ordinate proportional to the density at $M$. Then, since $Sm \times SM = SA^2 = Sf \times SF$, we have $Sm : Sf :: SF : SM$; and dividendo, $Sf - Sm : Sf :: SM - SF : SM$; alternando, $fm : FM :: Sf : SM :: Sf : SF :: SA^2 : SF^2$; $\therefore fm = \frac{FM \times SA^2}{SF^2}$; hence $fm \times fg$ varies as the fluxion of the distance × the force of gravity at $F \times$ the density, or the area $fgnm$ varies as the pressure at $F$; and the sum of all the similar areas below $fg$ varies as the sum of the pressures above $F$, or varies as the density $fg$. In the same manner the sum of all the areas below $mn$ varies as $mn$; hence their difference, or the area $fgnm$, $\propto fg - nm$; that is, $fg \times fm \propto fg - mn$; $\therefore$ if $fg = y$, and $fm = \dot{x}$, and $fg - mn = \dot{y}$, $y \times \dot{x} \propto \dot{y}$, or $\dot{x} \propto \frac{\dot{y}}{y}$, a property of the logarithmic curve. This curve is the same with that in the last Proposition; for the ordinates very near $AB$, and at very small equal intervals, are in each case equal; hence, in both cases, the curvature, the inclination of the tangent at $B$, and the value of the sub-tangent, are the same.

### PROB. 113.

The force of gravity being supposed to vary as the $n^{\text{th}}$ power of the distance from the center of the earth, and the compressive force of the air to be proportional to its density; to find the density of the air at any altitude above the surface.

Here the force of gravity, at a distance $x$ from the center, $= \frac{x^n}{r^n}$; therefore, on the same assumption as in the preceding case, $h : 1 :: \frac{vx^n\dot{x}}{r^n} : -\dot{v}$; $\therefore\ -h \times \frac{\dot{v}}{v} = \frac{x^n\dot{x}}{r^n}$, and $-h \times$ hyp. log. $v +$ corr. $= \frac{x^{n+1}}{\overline{n+1}\,.\,r^n}$ + corr. Now if $x = r$, $v = 1$; therefore $h \times$ hyp. log. $\frac{1}{v} = \frac{x^{n+1} - r^{n+1}}{\overline{n+1}\,.\,r^n}$.

COR. 1. If the force be constant, $n = 0$, and $h \times$ hyp. log. $\frac{1}{v} = x - r$, as in Prob. 111.

COR. 2. If the force $\propto \frac{1}{D^2}$ from the center, $n = -2$, and $h \times$ hyp. log. $\frac{1}{v} \propto 1 - \frac{r}{x}$, as in Prob. 112.

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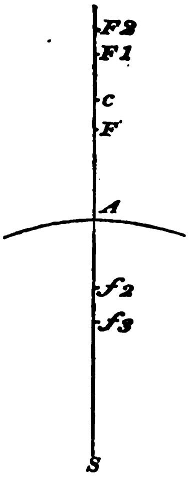
### COTES' SOLUTION.

Let $S$ be the center of the earth, $A$ a point on its surface, $SAF2$ a line drawn from the center to the top of the atmosphere; and let it be required to find the ratio of the density in $A$ to the density in $F$, the force of gravity varying as $SF^n$. Let $SF = x$, $d =$ the density at $A$, and r the density at $F$; then, since the compressive force varies as the density, the fluxion of the density is as the fluxion of the compressive force, or $\dot{v} \propto v x^n \dot{x}$, and $\frac{\dot{v}}{v} \propto x^n \dot{x}$. Let $AC$ be the height of an homogeneous atmosphere; that is, of an atmosphere, whose density throughout is the same with the density at $A$; then $AC$ : the height of the mercury in a barometer at $A$ :: the density of mercury : the density of the air at $A$; and if $F$ be conceived to approach toward $A$, the altitude of the mercury in the barometer at $A$ : its altitude at $F$ :: $AC$ : $FC$. Hence the density of the air ($d$) at $A$ : its density ($v$) at $F$ :: $AC$ : $FC$; $\therefore d-v$ or $\dot{v}$ : $d$ or $v$ :: $AF$ or $\dot{x}$ : $AC$; whence, in this case, $AC \times \frac{\dot{v}}{v} = \dot{x} = \frac{x^n \dot{x}}{SA^n}$. Since then, wherever $F$ be assumed, $\frac{\dot{v}}{v} \propto x^n \dot{x}$, $AC \times \frac{\dot{v}}{v}$ will $= \frac{x^n \dot{x}}{SA^n}$ in all cases, whatever be the position of $F$.



## EXAMPLES.

Ex. 1. Let the force vary as $\frac{1}{D}$ from the center.

Here $n = -1$, and $AC \times \frac{\dot{v}}{v} = SA \times \frac{\dot{x}}{x}$; $\therefore AC \times$ hyp. log. $v$ + corr. $= SA \times$ hyp. log. $x$ + corr.; that is, since $\dot{v}$ is negative when $\dot{x}$ is positive, and the density at $A = d$,

$$AC \times \text{hyp. log.} \frac{d}{v} = SA \times \text{hyp. log.} \frac{x}{r},$$

or the measure of the ratios between the densities $d$ and $v$ to the modulus $AC$ = the measure of the ratios between $SF$ and $SA$ to the modulus $SA$.

Ex. 2. Let the force of gravity $\propto D^n$ from the center.

In this case, by taking and correcting the fluents,

$$AC \times \text{hyp. log.}\ \frac{d}{v} = \frac{1}{n+1} \times : \frac{x^{n+1}}{SA^n} - SA,$$

the measure of the ratio between the densities $d$ and $v$ to the modulus $AC$.

Assume $SA$, $SF$, $SF1$, $SF2$, in geometric progression increasing; and $SF$, $SA$, $Sf2$, $Sf3$, in the same progression decreasing; and let the force of gravity vary as $\frac{1}{D^3}$; the measure of the ratio between the densities at $A$ and $F$ to the modulus $AC$, or $AC \times \text{hyp. log.}\ \frac{d}{v} = \frac{1}{2} SA - \frac{1}{2} \frac{SA^3}{SF^2}$.

Now since $SF : SA :: SA : Sf2$,

and $SA : Sf2$ (or $SF : SA$) $:: Sf2 : Sf3$,

we have, by the first proportion,

$$SF : Sf2 :: SF^2 : SA^2;$$

and by the second, $Sf2 : Sf3 :: SF : SA$;

$$\therefore\ SF : Sf3 :: SF^3 : SA^3;$$

hence $Sf3 = \frac{SA^3}{SF^2}$, and $\frac{1}{2} SA - \frac{1}{2} \frac{SA^3}{SF^2} = \frac{1}{2} SA - \frac{1}{2} Sf3 = \frac{1}{2} Af3$;

$\therefore\ AC \times \text{hyp. log.}\ \frac{d}{v} = \frac{1}{2} Af3$. In the same manner, if the force of gravity $\propto \frac{1}{D^2}$, $AC \times \text{hyp. log.}\ \frac{d}{v} = Af2$; if gravity be uniform, it $= AF$; if gravity vary directly as the distance, it $= \frac{1}{2} AF1$; if gravity vary as $D^2$, it $= \frac{1}{3} AF^2$, and so on in infinitum.

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# *The Reader is desired to make the following Corrections.*

PAGE 7. line 3. Insert $2xx'+x'^{2}$.
34. Figure. $m$ is in $En$ produced.
44. line 9 from the bottom. for *PFHR*, read *PFpr*.
52. 19. for "∴ a quadrant," read "∴ the arc of a quadrant."
54. *F* is omitted in Fig. 1.
66. 4th and 5th lines from bottom. for *SA*, read *Sa*.
83. line 2. for *TRS*, read *YRS*.
102. 12. for "circle," read "center."
112. 14. for *BCR*, read *BSR*.
115. 4 from bottom. for $x\times\sqrt{ax-x^2}$, read $\frac{x\times\sqrt{ax-x^2}}{2}$.
120. 4 from bottom. for $x+a\times\sqrt{2ax+x^2}$, read $x+a+\sqrt{2ax+x^2}$.
122. 4. for $\dot{\dot{y}}$, read $-\dot{\dot{y}}$.
140. 3. for ,43424968, read ,43429448.
142. 4. for $a\bar{x}$, read $a\dot{x}$.
145. *B* is omitted in the Figure.
155. 2. for $=$, read $\propto$.
174. 6 from bottom. after "$\sqrt{AC^2+BC^2}$," insert "where *BC* is the semi-minor axis."
184. 2. for *SPr*, read *SPR*, *SPr*.
194. 2 from bottom. after $T=\frac{8nd}{3\times\sqrt{4ma}}$, read $\times e^{\frac{3x}{8nd}}$.
203. bottom line. for $v-c$, read $c-v$.
215. line 6. for $x=a$, read $x=0$.
219. line 1 from bottom. for $\frac{ar^2+ry^2}{ar^2+\overline{a+r}\cdot y^2}$, read $\frac{ar^2+ry^2}{ar^2+\overline{a+r}\cdot y^2}\times y\dot{y}$.
222. 6 and 7. for "$x+$" and "$-\frac{1}{2}a^2+$," read "$x\times$" and "$-\frac{1}{2}a^2\times$."
229. bottom line. for $\frac{r^1}{4}$, read $\frac{r^2}{4}$.
232. line 5. for "70," read "74."
233. bottom line. for $z^n\dot{z}$, read $z^{n-1}\dot{z}$.
257. lines 6 and 13. for "Art. 129," read "Art. 135."
Ib. line 12. for $2^{\frac{5}{7}}$, read $2^{\frac{1}{7}}$.
258. lines 4 and 5. Insert "2" in the denominator of the third term of the series.

# ERRATA.

Page 269. line 12. for $\frac{t}{m}$, read $t$.
272. 5. for $1 \pm x$. read $1 \pm x^n$.
274. 10. for $\frac{x\dot{x}}{\sqrt{1-x^2}}$, read $\frac{x\dot{x}}{1-x^2}$.
275. lines 5, 6 and 7. Multiply 3 into the numerator of the third term of the series.
285. last line but one. for $nx^n y^n$, read $x^n y^n$.
Ib. last line. read $nax^{n+1}$.
288. line 2. denominator. for $x^m$, read $x^n$.
294. 1. for "chord," read "cord."
336. 11. for $\frac{2r\dot{y}}{\sqrt{r^2-y^2}}$, read $\frac{2r\dot{y}}{r^2-y^2}$.
351. 6. denominator. for $n-2n$, read $n.2n$.
352. 6. for $x^3\dot{x}$, read $x^2\dot{x}$.
352. Figure. omit the tangent $PL$.
359. line 15. for $SP^2-SY^2$, read $\sqrt{SP^2-SY^2}$.

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